Menemui Matematik (Discovering Mathematics) 46(3) (2024) 20-29

Menemui Matematik (Discovering Mathematics)

journal homepage: https://myjms.mohe.gov.my/index.php/dismath/

Some Cases on The Diophantine Equation $p^x + q^y = z^2$

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Received: 5 September 2024

Accepted: 20 November 2024

ABSTRACT

The purpose of this research is to find an integral solution for some cases on the Diophantine equation $p^x + q^y = z^2$. Four equations under consideration are $3^x + 16^y = z^2$, $3^x + 21^y = z^2$, $7^x + 13^y = z^2$, and $7^x + 32^y = z^2$ in which *x*, *y* and *z* are non-negative integers. The integral solutions to the Diophantine equation $3^x + 16^y = z^2$, are (1,0,2) and (2,1,5). Diophantine equations $3^x + 21^y = z^2$, and $7^x + 32^y = z^2$ have the solution of (x, y, z) is (1,0,2) and (2,1,9) respectively while Diophantine equation $7^x + 13^y = z^2$ has no integral solution.

Keywords: Diophantine Equations, Integer solutions.

INTRODUCTION

The Diophantine equation is one of the major scopes in the study of number theory. It has been defined as a polynomial equation consisting of two or more unknowns. In this study, we are only interested in finding an integral solution which is all the unknowns take only non-negative integer values.

Acu (2007) proved that the Diophantine equation $2^x + 5^y = z^2$ where *x*, *y* and *z* are non-negative integers has only two integral solutions $(3, 0, 3)$ and $(2, 1, 3)$. Suvarnamani et al. (2011) studied the Diophantine equation $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ has no non-negative integer solution.

In the following year, Rabago (2012) found that the Diophantine equation $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ have exactly two solutions (x, y, z) in non-negative integers. The solutions are $(1, 0, 2), (4, 1, 10)$ and $(1, 0, 2), (2, 1, 10)$ respectively.

Sroysang (2012b) proved that the Diophantine equation $3^x + 5^y = z^2$ has a unique solution $(1, 0, 2)$. The proof was shown by applying Catalan's conjecture. Catalan stated that the unique solution for the Diophantine equation $a^x - b^y = 1$ where *a*, *b*, *x*, and *y* are $(3, 2, 2, 3)$ with the minimum values $(a, b, x, y) > 1$. The Catalan's conjecture was proven by Mihailescu (2004).

There are a few research papers from Sroysang that applied Catalan's conjecture in order to prove the solutions for the particular Diophantine equation and we list it down as follows:

i) Sroysang (2012a) found that the Diophantine equation $8^x + 19^y = z^2$ has a unique non-negative integer solution and the solution is $(0, 1, 3)$.

ii) Sroysang (2013a) proved that the Diophantine equation $2^x + 3^y = z^2$ has exactly three non-negative integer solutions: $(x, y, z) = (0, 1, 2), (3, 0, 3), (4, 2, 5)$.

iii) Sroysang (2013b) proved that the Diophantine equation $7^x + 8^y = z^2$ has a unique solution: $(x, y, z) = (0, 1, 3)$.

iv) Sroysang (2014a) found that the Diophantine equation $3^x + 85^y = z^2$ has a unique solution: $(x, y, z) = (1, 0, 2)$.

v) Sroysang (2014b) found that the Diophantine equation $5^x + 43^y = z^2$ has no non-negative integer solution.

vi) Sroysang (2014c) proved that the Diophantine equation $5^x + 63^y = z^2$ has a unique solution $(x, y, z) = (0, 1, 8)$.

vii) Sroysang (2014d) proved the Diophantine equation $7^x + 31^y = z^2$ has no non-negative integer solution.

Terai and Hibino (2015) proved that the Diophantine equation $(12m^2 + 1)^x + (13m^2 - 1)^y = (5m)^2$ has only one non-negative integer solution that is $(x, y, z) = (1, 1, 2)$. In the following year, Rabago (2016) found that the Diophantine equation $2^x + 17^y = z^2$ has exactly five non-negative integer solutions and the solutions are $(3, 1, 5), (5, 1, 7), (6, 1, 9), (7, 3, 71)$ and $(9, 1, 23)$. Asthana and Singh (2017) showed that the Diophantine equation $3^x + 13^y = z^2$ has exactly four non-negative integer solutions and the solutions are $(1, 0, 2), (1, 1, 4), (3, 2, 14)$ and $(5, 1, 16)$. Burshtein (2019) found that the Diophantine equation $5^x + 103^y = z^2$ and $5^x + 11^y = z^2$ have no non-negative integer solution. Recently, Orosram and Comemuang (2020) proved that the Diophantine equation $8^x + n^y = z^2$ has only one non-negative integer solution that is $(x, y, z) = (1, 0, 3)$ where *x*, *y* and *z* are non-negative integers and $n \equiv 10 (mod\ 15)$.

SOME CASES ON THE DIOPHANTINE EQUATION

In this section, we will find an integral solution to the Diophantine equation $3^x + 16^y = z^2$, $3^x + 21^y = z^2$, $7^x + 13^y = z^2$, and $7^x + 32^y = z^2$ where *x*, *y*, and *z* are non-negative integers. To find the integral solution (*x, y, z*) for all cases that we consider, we highlight the Catalan's Conjecture that has been proved by Mihailescu (2004) as follows.

Proposition 1 (The Catalan's Conjecture)**.** *The unique solution for the Diophantine equation* $a^x - b^y = 1$ *where* $a, b, x,$ *and* y *is* $(3, 2, 2, 3)$ *with the minimum values* $(a, b, x, y) > 1$.

Now, we highlight our result for some cases on the Diophantine equation.

DIOPHANTINE EQUATION $3^x + 16^y = z^2$

In this subsection, we will find the integral solution to the Diophantine equation

$$3^x + 16^y = z^2. \quad (1)$$

In the following lemma, we consider the case $y = 0$ of the Diophantine equation $3^x + 16^y = z^2$.

Lemma 1. *The unique solution for the Diophantine equation*

$$3^x + 1 = z^2, \tag{2}$$

which is denoted as (x, z) *is* $(1, 2)$.

Proof: Assume $3^x + 1 = z^2$ has a solution such that x and z are non-negative integers. We will consider two cases based on the possible values of x.

Case 1: Let $x = 0$.
Equation (2) will become $z^2 = 3^0 + 1$. Then,

$$z^2 = 2.$$

Therefore $z = \sqrt{2}$ is contradiction since z is non-negative integer.

Case 2 : Let $x \geq 1$.
Firstly, we consider $x = 1$, thus the equation (2) will become

$$z^2 = 3^1 + 1$$
$$z = \pm 2,$$

since z is non-negative integer, thus we only choose $z = 2$.

Secondly, we consider $x > 1$. From Proposition 1, x cannot be greater than 1. Based on both cases, we conclude that the only possible solution for equation $3^x + 1 = z^2$ is $(x, z) = (1, 2)$.

■

In the following lemma, we consider the case $x = 0$ for the Diophantine equation $3^x + 16^y = z^2$.

Lemma 2. *Diophantine equation*

$$1 + 16^y = z^2 \tag{3}$$

has no non-negative integer solution.

Proof: Assume there exists the value of y and z which are non-negative integers. We will consider two cases based on the possible values of y.

Case 1: Let $y = 0$.
Equation (3) will become

$$z^2 = 1 + 16^0$$
$$z^2 = 2.$$

This is contradiction since z is a non-negative integer.

Case 2: Let $y \geq 1$.
Firstly, we consider $y = 1$, thus the equation (3) will become

$$z^2 = 1 + 16^1$$
$$= 17.$$

This is contradiction since z is a non-negative integer.

Secondly, we consider $y > 1$. From Proposition 1, y cannot be greater than 1. Thus, from the above two cases, the Diophantine equation $1 + 16^y = z^2$ has no non-negative integer solution.

■

Now, we highlight our results for the Diophantine equation $3^x + 16^y = z^2$ as in the following theorem.

Theorem 1. *The Diophantine equation*

$$3^x + 16^y = z^2, \tag{4}$$

has two solutions which are $(x, y, z) = (1, 0, 2)$ *and* $(2, 1, 5)$ *where x, y, and z are non-negative integers.*

Proof: Let x, y and z be non-negative integers such that $3^x + 16^y = z^2$ and consider the case when one of the three unknowns x, y, and z is zero.

If $y = 0$ then we can use Lemma 1 to conclude that the solution for equation $3^x + 1 = z^2$ is $(x, z) = (1, 2)$.

If $x = 0$ then we can use Lemma 2 to conclude that there is no non-negative integer solution for equation $1 + 16^y = z^2$.

Now, we will consider the case $x \geq 1$ and $y \geq 1$. Note that z is odd, it implies $z^2 \equiv 1(mod\ 4)$. Then we substitute $z^2 \equiv 1(mod\ 4)$ and $16^y \equiv 0(mod\ 4)$ into equation (4) and we have

$$\begin{aligned} 3^x &= z^2 - 16y \\ &= 1(mod\ 4) - 0(mod\ 4) \\ &= 1(mod\ 4). \end{aligned}$$

We obtain $3^x \equiv 1(mod\ 4)$. Note that, if $3^x \equiv 1(mod\ 4)$, it implies x is even. While if $3^x \equiv 3(mod\ 4)$, it implies x is odd. Therefore, x is even. Let $x = 2k$ where k is a positive integer. We substitute $x = 2k$ into equation (4) and we have

$$\begin{aligned} z^2 &= 3^{2k} + 16^y \\ z^2 &= 3^{2k} + 2^{4y} \\ 2^{4y} &= z^2 - 3^{2k} \\ 2^{4y} &= (z - 3^k)(z + 3^k). \end{aligned} \tag{5}$$

Let $z - 3^k = 2^u$ where u is a non-negative integer, therefore $z = 2^u + 3^k$ and we substitute into equation (5). This implies $2^{4y} = 2^u(z + 3^k)$. and we obtain,

$$\begin{aligned} 2^u(z + 3^k) &= 2^{4y} \\ 3^k &= \frac{2^{4y}}{2^u} - z \\ 3^k &= \frac{2^{4y}}{2^u} - (2^u + 3^k) \\ 2(3^k) &= \frac{2^{4y}}{2^u} - 2^u \\ 2(3^k) &= 2^u(2^{4y-2u} - 1). \end{aligned}$$

It follows that $u = 1$ and $3^k = 2^{4y-2} - 1$. Thus, $2^{4y-2} - 3^k = 1$. Since $k \geq 1$, we have $4y - 2 \geq 3$. By Proposition 1, we have $k = 1$. Then we obtain,

$$2^{4y-2} = 2^2.$$

Thus,

$$4y - 2 = 2$$
$$y = 1.$$

Since $x = 2k$ and $k = 1$, then $x = 2$ and $y = 1$, thus $z = 5$. Therefore, $(1, 0, 2)$ and $(2, 1, 5)$ are the solutions (x, y, z) for the equation $3^x + 16^y = z^2$ where x, y and z are non-negative integers.

■

Corollary 1. *The Diophantine equation $3^x + 16^y = w^4$ where x, y and w are non-negative integers has no non-negative integer solution.*

Proof: Suppose that there are non-negative integers x, y and w such that $3^x + 16^y = w^4$. Let $z = w^2$, this implies $3^x + 16^y = z^2$. By Theorem 1 the equation $3^x + 16^y = z^2$ has two non-negative integer solutions $(1, 0, 2)$ and $(2, 1, 5)$. Since we know that $z = 2$ and $z = 5$, thus $w = \sqrt{2}$ and $w = \sqrt{5}$. This is contradiction since w is non-negative integer. Therefore, it is proven that Diophantine equation $3^x + 16^y = w^4$ has no non-negative integer solution.

■

DIOPHANTINE EQUATION $3^x + 21^y = z^2$

In this section, we will find the integral solution to the Diophantine equation

$$3^x + 21^y = z^2. \tag{6}$$

Here, we consider the case $y = 0$ of the Diophantine equation $3^x + 21^y = z^2$.

Lemma 3. *The unique solution for the Diophantine equation*

$$3^x + 1 = z^2, \tag{7}$$

which is denoted as (x, z) is $(1, 2)$.

Proof for Lemma 3 can be handled similarly with Lemma 1.

In the following Lemma, we consider for the case $x = 0$ of the Diophantine equation $3^x + 21^y = z^2$.

Lemma 4. *Diophantine equation*

$$1 + 21^y = z^2, \tag{8}$$

has no non-negative integer solution.

Proof for Lemma 4 can be handled similarly with Lemma 2.

Now, we highlight our results for Diophantine equation $3^x + 21^y = z^2$ as in the following theorem.

Theorem 2. *The Diophantine equation*

$$3^x + 21^y = z^2, \tag{9}$$

has a unique solution which is $(x, y, z) = (1, 0, 2)$ *where* $x, y,$ *and* z *are non-negative integers.*

Proof: Assume x, y, and z are non-negative integers such that $3^x + 21^y = z^2$. Since the value of z is even, it implies $z^2 \equiv 0 (mod\ 4)$. Since $21 \equiv 1 (mod\ 4)$, then $21^y \equiv 1\ (mod\ 4)$.

Then, substitute $z^2 \equiv 0(mod\ 4)$ and $21^y \equiv 1\ (mod\ 4)$ into equation (9). Then we have

$$\begin{aligned} 3^x &= z^2 - 21^y \\ 3^x &\equiv 0(mod\ 4) - 1(mod\ 4) \\ &\equiv -1(mod\ 4) \\ &\equiv 3(mod\ 4). \end{aligned}$$

Note that, if $3^x \equiv 1\ (mod\ 4)$, it implies x is even. While if $3^x \equiv 3\ (mod\ 4)$, it implies x is odd. Thus, x is an odd number.

Now we will consider y into two cases.

Case 1 : Let $y = 0$. Equation (9) will become

$$3^x + 1 = z^2.$$

From Lemma 3, we have $x = 1$ and $z = 2$.

Case 2: Let $y \geq 1$. Firstly, we consider $y = 1$. It is clear that $21^y \equiv 1\ (mod\ 5)$. Since x is odd, then $3^x \equiv 2\ (mod\ 5)$ or $3^x \equiv 3(mod\ 5)$. By substituting $21^y \equiv 1(mod\ 5)$ and $3^x \equiv 2\ (mod\ 5)$ into equation $z^2 = 3x + 21$,we obtain

$$\begin{aligned} z^2 &= 3^x + 21^y \\ &\equiv 2(mod\ 5) + 1(mod\ 5) \\ &\equiv 3(mod\ 5). \end{aligned}$$

By substituting $21^y \equiv 1(mod\ 5)$ and $3^x \equiv 3\ (mod\ 5)$ into equation $z^2 = 3^x + 21$,we get

$$\begin{aligned} z^2 &= 3^x + 21^y \\ &\equiv 3(mod\ 5) + 1(mod\ 5) \\ &\equiv 4(mod\ 5). \end{aligned}$$

Thus, we have $z^2 \equiv 3, 4\ (mod\ 5)$. This is contradiction since z is even and $z^2 \equiv 0, 1, 4\ (mod\ 5)$.

Now, we consider $y > 1$. From Proposition 1, y cannot be greater than 1. Therefore, it is proven that $(1, 0, 2)$ is a unique solution to the Diophantine equation $3^x + 21^y = z^2$. ∎

Corollary 2 *The Diophantine equation* $3^x + 21^y = w^4$ *has no non-negative integer solution where x, y and w are non-negative integers.*

Proof: Suppose that there are non-negative integers x, y and w such that $3^x + 21^y = w^4$. Let $z = w^2$, this implies $3^x + 21^y = z^2$. By Theorem 2 the equation $3^x + 21^y = z^2$ has a unique solution

$(1,0,2)$. Since we know that $z=2$, thus $w=\sqrt{2}$. This is contradiction since w is non-negative integer. Therefore, it is proven that Diophantine equation $3^x+21^y=w^4$ has no non-negative integer solution.

■

DIOPHANTINE EQUATION $7^x+13^y=z^2$

In this section, we will prove that the Diophantine equation

$$7^x+13^y=z^2, \tag{10}$$

has no non-negative integer solution. In the following Lemma, we consider for the case $y=0$ of the Diophantine equation $7^x+13^y=z^2$.

Lemma 5. *The Diophantine equation*

$$7^x+1=z^2, \tag{11}$$

has no non-negative integer solution.

Proof for Lemma 5 can be handled similarly with Lemma 1.

In the following Lemma, we consider for the case $x=0$ of the Diophantine equation $7^x+13^y=z^2$.

Lemma 6. *Diophantine equation*

$$1+13^y=z^2, \tag{12}$$

has no non-negative integer solution.

Proof for Lemma 6 can be handled similarly with Lemma 2.

Now, we highlight our results for Diophantine equation $7^x+13^y=z^2$ as in the following theorem.

Theorem 3. *There is no non-negative integer solution for Diophantine equation*

$$7^x+13^y=z^2. \tag{13}$$

Proof: Suppose that there are non-negative integers solution *x, y* and *z such* that $7^x+13^y=z^2$. From Lemma 5 and 6, we obtain that $x\geq 1$ and $y\geq 1$. This implies that *z* is even. It follows that $z^2\equiv 0\ (mod\ 3)$ or $z^2\equiv 1\ (mod\ 3)$.

By substituting $7^x\equiv 1\ (mod\ 3)$ and $13^x\equiv 1\ (mod\ 3)$ into equation (13), we obtain

$$\begin{aligned} z^2 &= 7^x+13^y \\ &\equiv 1(mod\ 3)\ +\ 1(mod\ 3) \\ &\equiv 2(mod\ 3). \end{aligned}$$

This is contradiction because as z is even and $z^2 \equiv 0, 1 \ (mod\ 3)$.

Therefore, it is proven that Diophantine equation $7^x + 13^y = z^2$ has no non-negative integer solution.

■

DIOPHANTINE EQUATION $7^x + 32^y = z^2$

In this section, we will give integral solutions to the Diophantine equation

$$7^x + 32^y = z^2. \tag{14}$$

To solve the Diophantine equation (14), we need the following lemmas.

Lemma 7. *The Diophantine equation*

$$7^x + 1 = z^2, \tag{15}$$

has no non-negative integer solution.

Proof for Lemma 7 can be handled similarly with Lemma 1.

Lemma 8. *Diophantine equation*

$$1 + 32^y = z^2, \tag{16}$$

has no non-negative integer solution.

Proof for Lemma 8 can be handled similarly with Lemma 2.

Now, we highlight our results for Diophantine equation $7^x + 32^y = z^2$ as in the following theorem.

Theorem 4. *The Diophantine equation*

$$7^x + 32^y = z^2, \tag{17}$$

has a unique solution which is $(x, y, z) = (2, 1, 9)$ *where* $x, y,$ *and* z *are non-negative integers.*

Proof: Let x, y and z be non-negative integers such that $7^x + 32^y = z^2$ and consider the case when one of the three unknowns x, y, and z is zero. If $y = 0$ then we can use Lemma 2.7 to conclude that there is no non-negative integer solution for equation $7^x + 1 = z^2$. If $x = 0$ then we can use Lemma 8 to conclude that there is no non-negative integer solution for equation $1 + 32^y = z^2$.

Now, we will consider the case $x \geq 1$ and $y \geq 1$. Note that z is odd. Then $z^2 \equiv 1 \ (mod\ 4)$ and $32^y \equiv 0 (mod\ 4)$. Substitute $z^2 \equiv 1$ (mod 4) and $32^y \equiv 0$ (mod 4) into Equation (17), we have

$$\begin{aligned} 7^x &= z^2 - 32^y \\ &\equiv 1 (mod\ 4) + 0 (mod\ 4) \\ &\equiv 1 (mod\ 4). \end{aligned}$$

We obtain $7^x \equiv 1 \ (mod\ 4)$. This implies that x is even. Let $x = 2k$ where k is a positive integer. We substitute $x = 2k$ into equation (17) and we obtain

$$
\begin{aligned}
z^2 &= 7^{2k} + 32^y \\
z^2 &= 7^{2k} + 2^{5y} \\
2^{5y} &= z^2 - 7^{2k} \\
&= (z - 7^k)(z + 7^k). \qquad (18)
\end{aligned}
$$

Let $z - 7^k = 2^u$, then equation (18) will become

$$
\begin{aligned}
2^u(z + 7^k) &= 2^{5y} \\
7^k &= \frac{2^{5y}}{2^u} - z \\
7^k &= \frac{2^{5y}}{2^u} - (2^u + 7^k) \\
2(7^k) &= \frac{2^{5y}}{2^u} - 2^u \\
2(7^k) &= 2^u(2^{5y-2u} - 1).
\end{aligned}
$$

It follows that $u = 1$ and $7^k = 2^{5y-2} - 1$. Thus, $2^{5y-2} - 7^k = 1$. Since $k \geq 1$, we have $5y - 2 \geq 3$. By Proposition 1, we have $k = 1$. Then we obtain,

$$
\begin{aligned}
2^{5y-2} &= 1 + 7^1 \\
2^{5y-2} &= 2^3 \\
5y - 2 &= 3 \\
y &= 1.
\end{aligned}
$$

Since $x = 2k$ and $k = 1$, we have $x = 2$ and $y = 1$, thus $z = 9$. Therefore, $(2, 1, 9)$ is a unique solution (x, y, z) for the equation $7^x + 32^y = z^2$ where x, y and z are non-negative integers.

■

CONCLUSION

The integral solutions to the Diophantine equation $3^x + 16^y = z^2$, are (1,0,2) and (2,1,5). Diophantine equations $3^x + 21^y = z^2$, and $7^x + 32^y = z^2$ have the solution of (1,0,2) and (2,1,9) respectively while Diophantine equation $7^x + 13^y = z^2$ has no integral solution.

Acknowledgement

This research was funded by GERAN PUTRA 2021(GP-Matching UPM-UPNM/2021/9300477).

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