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Solving Second Order Delay Differential Equations of Constant Type Using Direct Implicit Method

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ABSTRACT

This paper will focus on the direct Adams-Moulton two-step method of order three (DAM2SM3) to solve second-order delay differential equations (DDEs) problems directly without transforming the equations into a first-order DDEs equation. This method is derived using Lagrange interpolation polynomials, followed by comprehensive analysis covering order, zero-stability, convergence, consistency and stability region in detail. Implementation is achieved through a C program featuring a predictor-corrector (PECE) scheme. Numerical results obtained indicate that the proposed direct method, DAM2SM3 is suitable for solving second-order delay differential equations.

Keywords: delay differential equations; direct method; implicit method; numerical simulation

INTRODUCTION

Delay differential equations (DDEs) are one of the types of differential equations that incorporate delayed information into the model. Nowadays, mathematical models using the concept of DDEs are developing rapidly and are useful for various real-life problems, especially in the field of science. For example, population dynamics, immunology, physiology, epidemiology and neural networks (Rihan et al. (2018)). Based on Kuang (1993), even a small delay can greatly affect the solution. Thus, considering delays in the mathematical model when finding a solution is an essential approach to obtaining more accurate results.

The general form of second-order DDEs for constant delay type is as follows.

$$\begin{gathered} y'' = f\big(t, y(t), y(t-\tau)\big), \qquad a \le t \le b, \qquad \tau > 0, \\ y'(a) = \Omega, \\ y(t) = \phi(t), \alpha \le t \le a, 0 \le \tau \le |a - \alpha|, \end{gathered} \tag{1}$$

where $\phi(t)$ is the initial function and τ is the delay term.

The functions DDEs are a type of differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. Therefore, this DDEs is different from ordinary differential equations (ODEs), where in ODEs, the solution is only obtained from the current situation without considering the historical conditions.

Over the years, researchers have utilized the Adams-Moulton method for solving numerical problems due to its capabilities. Majid et al. (2009) proposed a two-point two-step block method of variable step size in the simple form of the Adams-Moulton method for directly solving second-order ODEs, demonstrating its efficiency and stability. Anake (2011) developed a new class of one-step continuous implicit hybrid methods for solving initial value problems (IVPs) of general second-order ODEs, which offers high accuracy, a low error constant, a large absolute stability interval and exhibits zero-stability and convergence. Johari and Majid (2022) proposed a two-step Adams-Moulton method directly in predictor-corrector mode to solve second-order ODEs, with results indicating comparable error, fewer functions and faster execution than existing methods.

Furthermore, the contribution of studying solutions for second-order DDEs problem was started by Nevers and Schmitt (1971). In this paper, the researchers proposed a shooting method for boundary value problems, employing Euler's method to demonstrate problem-solving abilities. Radzi et al. (2012) proposed two and three-point one-step block methods to solve the DDEs. This work observed that the total number of steps and computational cost for the proposed method is reduced compared to the existing method. Hoo et al. (2013) addressed the direct Adams-Moulton method to solve second-order DDEs and concluded that the method demonstrates superiority in terms of accuracy and requires less computational cost.

Blanco-Cocom et al. (2012) proposed the Adomian decomposition method to approximate the solution of DDEs subject to history functions. The result shows that the proposed method works efficiently and accurately to DDEs problems. Meanwhile, Jaaffar et al. (2020) proposed a direct multistep block method to solve second-order DDEs with boundary conditions directly, demonstrating computational effectiveness in solving the second-order DDEs.

This paper aims to propose the direct method of Adams-Moulton of two-step with predictor-corrector scheme to solve the second-order DDEs directly and provide more accuracy result compared to previous method.

FORMULATION

Derivation of the Method

The direct Adams-Moulton method is derived through the integration process as much as twice over the interval $[t_n, t_{n+1}]$ and will use the Lagrange interpolation polynomial. In this project, only the corrector formula will be derived, while the predictor formula will use the existing formula from Majid et al. (2011). The process of derivation begins with the equations of ODEs since numerical methods for solving DDEs are commonly can be adapted from ODEs. Taking the general second-order ODEs as follows:

$$y'' = f(t, y, y').$$

Integrating the second-order ODEs once,

$$\int_{t_n}^{t_{n+1}} y''(t)dt = \int_{t_n}^{t_{n+1}} f(t,y,y')dt.$$

Therefore,

$$y'(t_{n+1}) - y'(t_n) = \int_{t_n}^{t_{n+1}} f(t,y,y')dt. \qquad (2)$$

Integrating the second-order ODEs twice,

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{t} y''(t)dtdt = \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} f(t,y,y')dtdt.$$

Therefore,

$$y(t_{n+1}) - y(t_n) - hy'(t_n) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} f(t,y,y')dtdt. \qquad (3)$$

Replacing $f(t,y,y')$ in equation (2) and (3) with the Lagrange interpolation polynomial by using three points, $\{(t_{n-1}, f_{n-1}), (t_n, f_n), (t_{n+1}, f_{n+1})\}$, since the method proposed is order three. Then, we have,

$$y'(t_{n+1}) - y'(t_n) = \int_{t_n}^{t_{n+1}} [\frac{(t-t_n)(t-t_{n+1})}{(t_{n-1}-t_n)(t_{n-1}-t_{n+1})} \; f_{n-1} + \frac{(t-t_{n-1})(t-t_{n+1})}{(t_n-t_{n-1})(t_n-t_{n+1})} \; f_m + \frac{(t-t_{n-1})(t-t_n)}{(t_{n+1}-t_{n-1})(t_{n+1}-t_n)} \; f_{n+1}]dt, \qquad (4)$$

$$y(t_{n+1}) - y(t_n) - hy'(t_n) = \int_{t_n}^{t_{n+1}} (t_{n+1}-t)[\frac{(t-t_n)(t-t_{n+1})}{(t_{n-1}-t_n)(t_{n-1}-t_{n+1})} \; f_{n-1} + \frac{(t-t_{n-1})(t-t_{n+1})}{(t_n-t_{n-1})(t_n-t_{n+1})} \; f_n + \frac{(t-t_{n-1})(t-t_n)}{(t_{n+1}-t_{n-1})(t_{n+1}-t_n)} \; f_{n+1} \;]dt \; . \qquad (5)$$

Take $s = \frac{t-t_{n+1}}{h}$ and replace $dt = hds$. After that evaluating and simplifying (4) and (5) from -1 to 0 to obtain the formula of point $y'(t_{n+1})$ by using MAPLE. Hence, the equation of the corrector formula for the direct Adams-Moulton method can be obtained from this derivation is as follows.

Corrector formulae:

$$y'_{n+1} = y'_n + \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1})$$
$$y_{n+1} = y_n + hy'_n + \frac{h^2}{24}(3f_{n+1} + 10f_n - f_{n-1}). \qquad (6)$$

The predictor formula equation referred to the existing article from Majid et al. (2011) is as shown below.

Predictor formulae:

$$y'_{n+1} = y'_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2})$$
$$y_{n+1} = y_n + hy'_n + \frac{h^2}{24}(19f_n - 10f_{n-1} + 3f_{n-2}). \qquad (7)$$

Order of the Method

The corrector formula as shown in (6) is written directly in the form of a matrix difference equation as follows:

$$\alpha Y_N = h\beta Y_N' + h^2\gamma F_N$$

where

$$Y_N = \begin{bmatrix} y(t_{n-1}) \\ y(t_n) \\ y(t_{n+1}) \end{bmatrix}, Y'_N = \begin{bmatrix} y'(t_{n-1}) \\ y'(t_n) \\ y'(t_{n+1}) \end{bmatrix}, F_N = \begin{bmatrix} f_{n-1} \\ f_n \\ f_{n+1} \end{bmatrix}.$$

Based on the corrector formula (6) that has been obtained, we can rewrite it in a matrix difference equation as,

$$0 = -y'_{n+1} + y'_n + \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1})$$
$$y_{n+1} - y_n = hy'_n + \frac{h^2}{24}(3f_{n+1} + 10f_n - f_{n-1}).$$

Then, rearrange it become,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y(t_{n-1}) \\ y(t_n) \\ y(t_{n+1}) \end{bmatrix} = h \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y'(t_{n-1}) \\ y'(t_n) \\ y'(t_{n+1}) \end{bmatrix} + h^2 \begin{bmatrix} -\frac{1}{12} & \frac{8}{12} & \frac{5}{12} \\ -\frac{1}{24} & \frac{10}{24} & \frac{3}{24} \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_n \\ f_{n+1} \end{bmatrix}. \quad (8)$$

Then, the coefficient value from the above matrix (8) will be substituted into the linear difference operator and the derivatives are using the Taylor expansion. The coefficient matrix, C_r is obtained as follows:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + a_k$$
$$C_1 = \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k - (\beta_0 + \beta_1 + \cdots + \beta_k)$$
$$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + \cdots + k^2\alpha_k) - (\beta_1 + 2\beta_2 + \cdots + k\beta_k) - (\gamma_0 + \gamma_1 + \cdots + \gamma_k)$$
$$\cdots$$
$$C_r = \sum_{j=0}^{k} \left(\frac{j^r}{r!}\alpha_j - \frac{j^{r-1}}{(r-1)!}\beta_j - \frac{j^{r-2}}{(r-2)!}\gamma_j \right) \text{ where } r = 3,4,5, \ldots.$$

By substituting the matrix difference equation into the formula above, we have

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_1 = \alpha_1 + 2\alpha_2 - (\beta_0 + \beta_1 + \beta_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_2 = \frac{\alpha_1}{2} + 2\alpha_2 - (\beta_1 + 2\beta_2) - (\gamma_0 + \gamma_1 + \gamma_2) = \frac{1}{2}\begin{bmatrix} 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_3 = \frac{\alpha_1}{6} + \frac{8\alpha_2}{6} - \left(\frac{\beta_1}{2} + 2\beta_2\right) - (\gamma_1 + 2\gamma_2) = \frac{1}{6}\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{8}{6}\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{18}{12} \\ \frac{16}{24} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_4 = \frac{\alpha_1}{24} + \frac{16\alpha_2}{24} - \left(\frac{\beta_1}{6} + \frac{8\beta_2}{6}\right) - \left(\frac{\gamma_1}{2} + 2\gamma_2\right) = \begin{bmatrix} 0 \\ \frac{15}{24} \end{bmatrix} - \begin{bmatrix} -\frac{7}{6} \\ \frac{1}{6} \end{bmatrix} - \begin{bmatrix} \frac{14}{12} \\ \frac{11}{24} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_5 = \frac{\alpha_1}{120} + \frac{32\alpha_2}{120} - \left(\frac{\beta_1}{24} + \frac{16\beta_2}{24}\right) - \left(\frac{\gamma_1}{6} + \frac{8\gamma_2}{6}\right) = \begin{bmatrix} 0 \\ \frac{31}{120} \end{bmatrix} - \begin{bmatrix} -\frac{15}{24} \\ \frac{1}{24} \end{bmatrix} - \begin{bmatrix} \frac{48}{72} \\ \frac{34}{144} \end{bmatrix} = \begin{bmatrix} -\frac{1}{24} \\ -\frac{7}{360} \end{bmatrix}.$$

Definition 2.1. (Lambert (1973)) The method is said to have an order m if $C_0 = C_1 = \cdots = C_m = C_{m+1} = 0$ and $C_{m+2} \neq 0$. The value of C_{m+2} is become the error constant.

Thus, the proposed method is on order three with the error constant $\begin{bmatrix} -\frac{1}{24} & -\frac{7}{360} \end{bmatrix}^T$ and the method is known as Direct Adams-Moulton two-step method of order three (DAM2SM3).

Zero Stability of the Method

To show zero stability of the method, we have the following definition.

Definition 2.2. (Lambert (1973)) If there is no root of the first characteristic polynomial $\rho(R)$ has a modulus larger than one, and all roots with modulus one are simple roots, the approach is said to have zero-stability.

This method is zero-stable when the root state R_j of the first characteristic polynomial $\rho(R)$is defined as

$$\rho(R) = \det\left[\sum_{j=0}^{k} A_{(j)}\, R^{k-j}\right] = \det[A_0 R - A_1] = 0,$$

which satisfies $|R_j| \leq 1$.

Based on the corrector formula equation (6), $A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\rho(N) = \det\begin{bmatrix} R-1 & 0 \\ 0 & R-1 \end{bmatrix} = 0,$$

$$(R-1)^2 = 0, \qquad R = 1,1.$$

Since $|R_j| \leq 1$, the method is said to be zero-stable.

Consistency and Convergence

Definition 2.3. (Lambert (1973)) The method is said to be consistent if it passes the condition where the order of the method, $m \geq 1$.

From definition 2.3, the method is consistent since the proposed method (DAM2SM3) is in order three.

Definition 2.4. (Lambert (1973)) The linear multistep method is convergent when it is zerostable and consistent.

Therefore, based on definition 2.4, the DAM2SM3 is said to be convergent since it is zero-stable, $|Rj| \leq 1$ and consistent because the method has order three with error constant

$$C_5 = \begin{bmatrix} -\frac{1}{24} & -\frac{7}{360} \end{bmatrix}^T.$$

Stability Region

The stability region will be obtained by substitute the test equation

$$y'' = f = \lambda y(t) + \mu y(t-\tau)$$

into the proposed method (6). We obtain as follows,

$$y'_{n+1} = y'_n + \frac{5}{12}h\lambda y'_{n+1} + \frac{5}{12}h\beta y_{n+1} + \frac{2}{3}h\lambda y'_n + \frac{2}{3}h\beta y_n - \frac{1}{12}h\lambda y'_{n-1} - \frac{1}{12}h\beta y_{n-1}.$$

$$y_{n+1} = y_n + hy'_n + \frac{1}{8}h^2\lambda y'_{n+1} + \frac{1}{8}h^2\beta y_{n+1} + \frac{5}{12}h^2\lambda y'_n + \frac{5}{12}h^2\beta y_n - \frac{1}{24}h^2\lambda y'_{n-1} - \frac{1}{24}h^2\beta y_{n-1}.$$

Then rearrange equations above in matrix form

$$\begin{bmatrix} 1 & -\frac{5}{12}h\lambda \\ 0 & 1-\frac{1}{8}h^2\lambda \end{bmatrix}\begin{bmatrix} y'_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y'_n \\ y_n \end{bmatrix} + h\begin{bmatrix} 0 & \frac{5}{12}\mu + \frac{2}{3}\lambda \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y'_n \\ y_n \end{bmatrix}$$
$$+h\begin{bmatrix} 0 & \frac{2}{3}\mu - \frac{1}{12}\lambda \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y'_{n-1} \\ y_{n-1} \end{bmatrix} + h\begin{bmatrix} 0 & -\frac{1}{12}\mu \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y'_{n-2} \\ y_{n-2} \end{bmatrix}$$
$$+h^2\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{8}\mu + \frac{5}{12}\lambda \end{bmatrix}\begin{bmatrix} y'_n \\ y_n \end{bmatrix} + h^2\begin{bmatrix} 0 & 0 \\ 0 & \frac{5}{12}\mu - \frac{1}{24}\lambda \end{bmatrix}\begin{bmatrix} y'_{n-1} \\ y_{n-1} \end{bmatrix} + h^2\begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{24}\mu \end{bmatrix}\begin{bmatrix} y'_{n-2} \\ y_{n-2} \end{bmatrix}.$$

From above matrix, we have

$$A_0 = \begin{bmatrix} 1 & -\frac{5}{12}h\lambda \\ 0 & 1-\frac{1}{8}h^2\lambda \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & \frac{5}{12}\mu + \frac{2}{3}\lambda \\ 1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & \frac{2}{3}\mu - \frac{1}{12}\lambda \\ 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & -\frac{1}{12}\mu \\ 0 & 0 \end{bmatrix},$$

$$C_1 = h^2 \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{8}\mu + \frac{5}{12}\lambda \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{5}{12}\mu - \frac{1}{24}\lambda \end{bmatrix}, C_3 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{24}\mu \end{bmatrix}.$$

General equation of stability is as follows,

$$\sum_{k=0}^{r} A_k Y_{m-k} + h \sum_{k=0}^{r+1} B_k Y_{m-k} + h^2 \sum_{k=0}^{r+1} C_k Y_{m-k} = 0.$$

By substituting the value $r = 1$, we have

$$\sum_{k=0}^{1} A_k Y_{m-k} + h \sum_{k=0}^{2} B_k Y_{m-k} + h^2 \sum_{k=0}^{2} C_k Y_{m-k} = 0.$$

Solving the determinant of

$$v^2(A_0 - hB_0 - h^2 C_0) - v(A_1 + hB_1 + h^2 C_1) - (hB_2 + h^2 C_2) = 0$$

By substitute $Y = h^2\lambda$ and $X = h^2\mu$, the stability polynomial is obtained

$$\frac{24v^6 - 3Yv^6 - 3Xv^5 - 17Yv^5 - 48v^5 - 17Xv^4 - 5Yv^4 + 24v^4 - 5Xv^3 + Yv^3 + Xv^2}{24} = 0. \quad (9)$$

The stability region is depicted in the $(X - Y)$ plane using the boundary locus technique, by substituting $v = 0, -1$, and $v = cos\,\theta + i\,sin\,\theta$, with $0 \leq \theta \leq 2\pi$, into the stability polynomial (9). By separating and solving the real and imaginary parts of $v = cos\,\theta + i\,sin\,\theta$ simultaneously, we determine the points within the region. As all roots of the stability polynomial (9) satisfy $|v| \leq 1$ and fall within the boundary of the region, the stability region is deemed stable. The shaded area in Figure 1 represents the stability region for DAM2SM3 for solving DDEs problem.

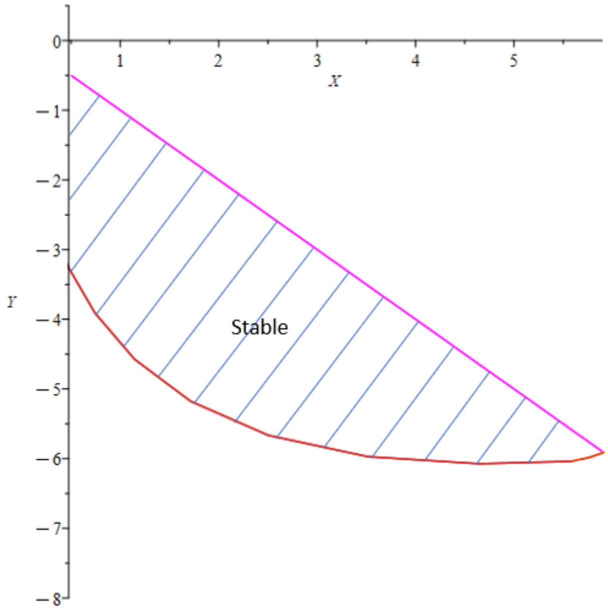

Figure 1: Stability region for DAM2SM3

IMPLEMENTATION

The implementation of DAM2SM3 method to solve second-order DDEs are in the form of predictor-corrector (PECE) scheme. Before applying DAM2SM3 method, the set of previous values are required first since the proposed method is a multistep method.

Therefore, three previous values including the initial value provided which are y_0, y_1, y_2 are needed and can be obtained by using the one-step method approach such as exact solution before proceeding to the proposed method. Once the previous values are calculated, DAM2SM3 method will be implemented as the corrector and the equation from Majid et al. (2011) will act as predictor.

For the constant delay type of problem, consequently for $(t - \tau) \leq a$, the delay term is calculated using the initial function, $\phi(t)$ to calculate $y(t - \tau)$ and $y'(t - \tau)$. Otherwise, for $(t - \tau) \geq a$, the delay term are depend on the location of $(t - \tau)$. From this location we are able to recall the values $y(t - \tau)$ and $y'(t - \tau)$ which we had stored earlier since the implementation in fixed step size i.e., $\tau = mh$ for $m = 1, 2, 3, \ldots$. In this project, the algorithm of the proposed method DAM2SM3 were developed in C language with the selection of step size is pre-determined.

Algorithm of DAM2SM3 Method

Step 1: Set starting value k, ending value l, step size h, given initial value and given initial function $\phi(t)$.
Step 2: For $n = 0, 1$, set $t_{n+1} = k + nh$, compute function f_n and delay term d_n. Evaluate y_{n+1} using exact solution.
Step 3: For $n \geq 2$, while $t_n < l$, do Steps 4 and 5.
Step 4: Set $t_{n+1} = t_n + h$, compute function f_n and delay term d_n. Calculate the approximate values of y'_{n+1} and y_{n+1} using the predictor formula.

Step 5: Calculate the approximate values of y'_{n+1} and y_{n+1} using the corrector formula.

Step 6: End.

Numerical Results

Three second-order DDEs problems with constant delay type were examined to assess the efficiency of the proposed DAM2SM3 method. These problems were tested using various step size, h, ranging from $h = 0.1$ to $h = 0.001$. The solutions obtained from the tested problems were compared between the DAM2SM3 method and an existing method, such as the direct Adams-Bashforth method of order three (DAB3). The following abbreviations are used in the tables which summarize the numerical results.

h Step size
FCN Total function calls
MAXE Maximum error
TIME Timing in second
DAM2SM3 Direct Adams-Moulton two-step method of order three
DAB3 Direct Adams-Bashforth of order three

Problem 1.

$$y''t) + y(t) = y(t - 1), t \in [0,1],$$
$$y(t) = t^2 + 3t + 2, \quad -1 \leq t \leq 0,$$
$$y'(0) = 0.$$

The exact solution:

$$y(t) = 4cos(t) - sin(t) + t^2 + t - 2.$$

Source: Martin and Garcia (2002).

Table 1: Solution of DAM2SM3 and DAB3 for Problem 1

h	METHOD	MAXE	FCN	TIME (s)
0.1	DAB3	2.48094290e-004	13	0.004251
	DAM2SM3	3.22901862e-005	21	0.006746
0.01	DAB3	3.69316455e-007	103	0.007393
	DAM2SM3	4.16524555e-008	201	0.016494
0.001	DAB3	3.82209819e-010	1003	0.136830
	DAM2SM3	4.25290914e-011	2001	0.152671

Problem 2.

$$y''(t) = -\frac{1}{2}y(t) + \frac{1}{2}y(t - \pi), t \in [0,1],$$
$$y(t) = 1 - sin(t), -\pi \leq t \leq 0.$$

The exact solution:

$$y(t) = 1 - sin(t).$$

Source: Rasdi et al. (2013).

Table 2: Solution of DAM2SM3 and DAB3 for Problem 2

h	METHOD	MAXE	FCN	TIME (s)
0.1	DAB3	1.01903807e-004	13	0.002005
	DAM2SM3	1.15489391e-005	21	0.005061
0.01	DAB3	1.58114680e-007	103	0.023840
	DAM2SM3	1.75833060e-008	201	0.033712
0.001	DAB3	1.64265962e-010	1003	0.078353
	DAM2SM3	1.82533155e-011	2001	0.088725

Problem 3.

$$y''(t) = y(t - \pi), t \in [0,1],$$
$$y(t) = sin(t), -\pi \leq t \leq 0.$$

The exact solution:

$$y(t) = sin(t).$$

Source: Rasdi et al. (2013).

Table 3: Solution of DAM2SM3 and DAB3 for Problem 3

h	METHOD	MAXE	FCN	TIME (s)
0.1	DAB3	1.04577206e-004	13	0.003890
	DAM2SM3	1.18034197e-005	21	0.004925
0.01	DAB3	1.64908750e-007	103	0.009505
	DAM2SM3	1.83415937e-008	201	0.016934
0.001	DAB3	1.71632264e-010	1003	0.095217
	DAM2SM3	1.90724103e-011	2001	0.121201

The performance of the proposed method, the direct Adams-Moulton two-step of order three (DAM2SM3), is discussed in detail with references to Tables 1 - 3 for each problem respectively. The results are compared with the DAB3 method in terms of maximum error, total function calls and execution time.

Based on Tables 1 - 3, we observed that the proposed DAM2SM3 method consistently shows good performance than DAB3 method as the step size decreases. Regarding the maximum error, DAM2SM3 method produces less error for each problem than DAB3. However, for the number of function calls and execution time, the DAM2SM3 method is slightly defeated by the DAB3 method. The results is reasonable since the DAB3 method only depends on the predictor scheme while DAM2SM3 uses the predictor-corrector scheme method. Therefore, the number of function calls and execution time for DAB3 is lower than the DAM2SM3. Overall, it has been proven that the DAM2SM3 method is comparable to existing methods and is effective in solving second-order DDE problems.

CONCLUSION

In this paper, the direct Adams-Moulton two-step method of order three (DAM2SM3) have been proposed to solve second-order delay differential equations (DDEs) problems. The proposed method has shown good performance and produces better results than the existing method DAB3 in terms of maximum error when solving the problems. Less in the maximum error, indicating that the results obtained are more accurate and closer to the exact solution. In conclusion, this proposed DAM2SM3 method is proven to be able to solve the second-order DDEs problems efficiently.

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