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On n-Algebras Arising From Binary Algebras

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ABSTRACT
This paper presents the definitions of n-algebra for $3 \leq n \leq 5$. It is known that the concept of linearity is important to gain the definition of algebra or binary algebra. This concept of binary algebra is used to introduce new definitions of n-algebra. As the implementation of these definitions, the classification of two dimensional complex n-Lie, n-associative and n-Leibniz algebras for $= 3, 4$ are provided.

Keywords: Lie algebra, associative algebra, Leibniz algebra, n-Lie algebras, n-algebras

INTRODUCTION

This section gives basic concepts of algebra that are applied in this study. It includes definitions of algebra and n-algebra where n is element of natural number, and also the classification of Lie, associative and Leibniz algebras in dimension two over complex field. Subsequently, there are literature reviews which are related to the study.

Ayupov et al. (2020) presents an algebra A as a vector space V over a field F equipped by a bilinear binary operation $f: V \times V \rightarrow V$ on it. The vector space V is called the underlying vector space of A. The definition of algebra can be written as:

Definition 1: A vector space V with bilinear operation $f: V \times V \rightarrow V$ satisfies these two conditions:

1. $f(\lambda_1 x_1 + \lambda_2 x_1', x_2) = \lambda_1 f(x_1, x_2) + \lambda_2 f(x_1', x_2)$

2. $f(x_1, \lambda_1 x_2 + \lambda_2 x_2') = \lambda_1 f(x_1, x_2) + \lambda_2 f(x_1, x_2')$

is called 2-algebra or algebra.

Note that 2-algebra or algebra is also called as binary algebra. Any field is an algebra over itself and over its subfield. The examples of algebra are the fields of rational number $\mathbb{Q}$, real number $\mathbb{R}$ and complex number $\mathbb{C}$. The set of polynomials $F[x_1, x_2, \cdots, x_n]$ at variables $x_1, x_2, \cdots, x_n$ with coefficients from a field F is also an algebra over F.

If the bilinear operation f in Definition 1 satisfies:

1. Jacobi identity, $f(f((x,y),z)+f(f(y,z),x)+f(f(z,x),y)=0$ and (skew) anti-symmetric $f(x,y)=-f(y,x)$, Lie algebra will obtained (Husain et al., 2017).

2. Associative identity $f(f(x,y),z)=f(x,f(y,z))$, an algebra becomes associative algebra (Rahman et al., 2021).

3. Leibniz identity, $f(x,f(y,z))=f(f(x,y),z)-f(f(x,z),y)$, an algebra will be called Leibniz algebra (Mohamed et al., 2020).

The classification of two-dimensional Lie, associative and Leibniz algebras over complex field are shown in Theorems 1, 2 and 3.

Theorem 1: (Ayupov et al., 2020) Two-dimensional Lie algebra is

$$L_1^1\colon [e_1e_2]=e_2,\,[e_2e_1]=-e_2.$$

Theorem 2: (Rahman et al., 2021) There are non-isomorphic of two-dimensional associative algebras

$$As_2^1\colon e_1e_1=e_2;\quad As_2^2\colon e_1e_1=e_1,\ e_1e_2=e_2;\quad As_2^3\colon e_1e_1=e_1,\ e_2e_1=e_2;$$

$$As_2^4\colon e_1e_1=e_1,\ e_2e_2=e_2;\quad As_2^5\colon e_1e_1=e_1,\,e_1e_2=e_2,\ e_2e_1=e_2.$$

Theorem 3: (Ayupov et al., 2020) Any two-dimensional Leibniz algebras is isomorphic to one of the following non-isomorphic Leibniz algebras

$$Lb_2^1\colon [e_1,e_1]=e_2;\ \ Lb_2^2\colon [e_1,e_2]=e_2,\ [e_2,e_1]=-e_2;\ \ Lb_2^3\colon [e_1,e_2]=e_1,\ [e_2,e_2]=e_1.$$

These classes of algebras will be used to find the classification of n-Leibniz, n-associative and n-Lie algebras for $n=3,4$ (see Propositions 1 until 9).

The definition of 3-algebra that involves trilinear operation is shown as follows:

Definition 2: A vector space V with trilinear operation $f\colon V\times V\times V\to V$ satisfies these three conditions:

1. $f(\lambda_1x_1+\lambda_2x_1',x_2,x_3)=\lambda_1f(x_1,x_2,x_3)+\lambda_2f(x_1',x_2,x_3)$

2. $f(x_1,\lambda_1x_2+\lambda_2x_2',x_3)=\lambda_1f(x_1,x_2,x_3)+\lambda_2f(x_1,x_2',x_3)$

3. $f(x_1,x_2,\lambda_1x_3+\lambda_2x_3')=\lambda_1f(x_1,x_2,x_3)+\lambda_2f(x_1,x_2,x_3')$

is called 3-algebra.

Observe from Definitions 1 and 2, it is clear that for n is natural number, n-algebra can be defined as in Definition 3 below:

Definition 3: A vector space V with multilinear map $f: V^{\otimes n} \to V$ satisfies the following conditions:

$$f(\lambda_1 x_1 + \lambda_2 x_1', x_2, \ldots, x_n) = \lambda_1 f(x_1, x_2, x_3) + \lambda_2 f(x_1', x_2, \ldots, x_n)$$

$$f(x_1, \lambda_1 x_2 + \lambda_2 x_2', \ldots, x_n) = \lambda_1 f(x_1, x_2, x_3) + \lambda_2 f(x_1, x_2', \ldots, x_n)$$

$$\vdots$$

$$f(x_1, x_2, \ldots, \lambda_1 x_n + \lambda_2 x_n') = \lambda_1 f(x_1, x_2, x_3) + \lambda_2 f(x_1, x_2, \ldots, x_n')$$

is called n-algebra.

One of the oldest branches of modern algebra is the theory of finite dimensional algebras. The work of Hamilton is the first who introduced the famous algebra of quaternion and matrix theory by Cayley. Besides that, B. Peirce, C.S. Peirce, Clifford, Weierstrass, Dedekind, Jordan and Frobenius are introduced finite dimensional algebras (Drozd and Kirichenko, 1991). In 1985, Filippov introduced the concept of n-Lie algebras and classified the $(n+1)$-dimensional n-Lie algebras over an algebraically closed field and characteristic zero. The structure of n-Lie algebras is very different from that of Lie algebras due to n-ary multilinear operation involved. The description of simultaneous classical dynamics of three particles is first appeared in Nambu's work by Bai et al. (2010). They give a complete isomorphism class n-Lie algebras over an algebraically closed field of characteristic zero in dimension $(n+1)$ and $(n+2)$ (Bai et al., 2011). A vector space equipped with an n-ary operation which has the property of being a derivation for itself is called a Leibniz n-algebra. The notion of Leibniz algebra is $n = 2$. Casas et al. (2002) described the free Leibniz $(n+1)$-algebra in the terms of the n-magma, that is the set of n-ary planar trees. Leibniz $(n+1)$-algebra into a Leibniz algebra is shown that the n-tensor power factor. This paper deals with n-algebra arising from algebra. We start by introducing n-algebra arising from algebra for $n = 3, 4, 5$. As implementation, the classifications of n-Lie, n-associative and n-Leibniz algebras for $n = 3, 4$ in dimension two are provided.

DEFINITIONS OF n-ALGEBRA ARISING FROM ALGEBRA

In this section, the definitions of n-algebra arising from binary algebra for $3 \leq n \leq 5$ are presented.

Consider V is a vector space over a field F. Given bilinear operation $\mu: V \times V \to V$, then by Definition 1, an algebra, (V, μ) is obtained.

Now, to find the definition of 3-algebra that arising from algebra, we create triple operation $f: V \times V \times V \to V$ defines by

$$f(x_1, x_2, x_3) = \mu\big(x_1, \mu(x_2, x_3)\big) \text{ and } f(x_1, x_2, x_3) = \mu(\mu(x_1, x_2), x_3).$$

From Definition 2 and equation $f(x_1, x_2, x_3) = \mu\big(x_1, \mu(x_2, x_3)\big)$, we can show that 3-algebra is arising from algebra as follows:

$$\begin{aligned} f(\lambda_1 x_1 + \lambda_2 x_1', x_2, x_3) &= \mu\big(\lambda_1 x_1 + \lambda_2 x_1', \mu(x_2, x_3)\big) \\ &= \lambda_1 \mu\big(x_1, \mu(x_2, x_3)\big) + \lambda_2 \mu\big(x_1', \mu(x_2, x_3)\big) \\ &= \lambda_1 f(x_1, x_2, x_3) + \lambda_2 f(x_1', x_2, x_3), \end{aligned}$$

$$\begin{aligned} f(x_1, \lambda_1 x_2 + \lambda_2 x_2', x_3) &= \mu\big(x_1, \mu(\lambda_1 x_2 + \lambda_2 x_2', x_3)\big) \\ &= \lambda_1 \mu\big(x_1, \mu(x_2, x_3)\big) + \lambda_2 \mu\big(x_1, \mu(x_2', x_3)\big) \\ &= \lambda_1 f(x_1, x_2, x_3) + \lambda_2 f(x_1, x_2', x_3), \end{aligned}$$

$$\begin{aligned} f(x_1, x_2, \lambda_1 x_3 + \lambda_2 x_3') &= \mu\big(x_1, \mu(x_2, \lambda_1 x_3 + \lambda_2 x_3')\big) \\ &= \lambda_1 \mu\big(x_1, \mu(x_2, x_3)\big) + \lambda_2 \mu\big(x_1, \mu(x_2, x_3')\big) \\ &= \lambda_1 f(x_1, x_2, x_3) + \lambda_2 f(x_1, x_2, x_3'). \end{aligned}$$

Since $f(x_1, x_2, x_3) = \mu\big(x_1, \mu(x_2, x_3)\big)$ satisfies the linearity then (V, f) is 3-algebra.

Similar method will be apply for equation $f(x_1, x_2, x_3) = \mu(\mu(x_1, x_2), x_3)$ to obtain 3-algebra which can be expressed from binary algebra shows as:

$$\begin{aligned} f(\lambda_1 x_1 + \lambda_2 x_1', x_2, x_3) &= \mu(\mu(\lambda_1 x_1 + \lambda_2 x_1', x_2), x_3) \\ &= \lambda_1 \mu(\mu(x_1, x_2), x_3) + \lambda_2 \mu(\mu(x_1', x_2), x_3) \\ &= \lambda_1 f(x_1, x_2, x_3) + \lambda_2 f(x_1', x_2, x_3), \end{aligned}$$

$$\begin{aligned} f(x_1, \lambda_1 x_2 + \lambda_2 x_2', x_3) &= \mu(\mu(x_1, \lambda_1 x_2 + \lambda_2 x_2'), x_3) \\ &= \lambda_1 \mu(\mu(x_1, x_2), x_3) + \lambda_2 \mu(\mu(x_1, x_2'), x_3) \\ &= \lambda_1 f(x_1, x_2, x_3) + \lambda_2 f(x_1, x_2', x_3), \end{aligned}$$

$$\begin{aligned} f(x_1, x_2, \lambda_1 x_3 + \lambda_2 x_3') &= \mu(\mu(x_1, x_2), \lambda_1 x_3 + \lambda_2 x_3') \\ &= \lambda_1 \mu(\mu(x_1, x_2), x_3) + \lambda_2 \mu(\mu(x_1, x_2), x_3') \\ &= \lambda_1 f(x_1, x_2, x_3) + \lambda_2 f(x_1, x_2, x_3'). \end{aligned}$$

Therefore (V, f) is 3-algebra since $f(x_1, x_2, x_3) = \mu(\mu(x_1, x_2), x_3)$ satisfies the linearity.

From the arguments above, 3-algebra that arises from algebra shows in Definition 4.

Definition 4: Suppose V is a vector space. Let (V, μ) be an algebra where $\mu: V \times V \to V$ is bilinear. Let trilinear operation $f: V \times V \times V \to V$ satisfies one of the following

$$f(x_1, x_2, x_3) = \mu\big(x_1, \mu(x_2, x_3)\big), \quad (1)$$

$$f(x_1, x_2, x_3) = \mu(\mu(x_1, x_2), x_3), \quad (2)$$

then (V, f) is called 3-algebra.

Now, use the similar method to introduce 4-algebra that arising from algebra. First, Let operations $\mu: V \times V \to V$ and $f: V \times V \times V \times V \to V$. Since μ is bilinear, from Definition 1, (V, μ) becomes an algebra. The operation f can be define as:

$$f(x_1, x_2, x_3, x_4) = \mu\big(\mu(x_1, x_2), \mu(x_3, x_4)\big),$$

$$f(x_1,x_2,x_3,x_4)=\mu(\mu(\mu(x_1,x_2),x_3),x_4),$$

$$f(x_1,x_2,x_3,x_4)=\mu\Big(x_1,\mu\big(x_2,\mu(x_3,x_4)\big)\Big),$$

$$f(x_1,x_2,x_3,x_4)=\mu\big(\mu\big(x_1,\mu(x_2,x_3)\big),x_4\big),$$

$$f(x_1,x_2,x_3,x_4)=\mu\big(x_1,\mu(\mu(x_2,x_3),x_4)\big).$$

Clear that Definition 3 for $n=4$ and equation $f(x_1,x_2,x_3,x_4)=\mu\big(\mu(x_1,x_2),\mu(x_3,x_4)\big)$ gives:

$$\begin{aligned}f(\lambda_1x_1+\lambda_2x_1',x_2,x_3,x_4)&=\mu\big(\mu\lambda_1x_1+\lambda_2x_1',x_2,\mu(x_3,x_4)\big)\\&=\mu\big(\lambda_1\mu(x_1,x_2)+\lambda_2\mu(x_1',x_2),\mu(x_3,x_4)\big)\\&=\lambda_1\mu\big(\mu(x_1,x_2),\mu(x_3,x_4)\big)+\lambda_2\mu\big(\mu(x_1',x_2),\mu(x_3,x_4)\big)\\&=\lambda_1f(x_1,x_2,x_3,x_4)+\lambda_2f(x_1',x_2,x_3,x_4),\end{aligned}$$

$$\begin{aligned}f(x_1,\lambda_1x_2+\lambda_2x_2',x_3,x_4)&=\mu\big(\mu(x_1,\lambda_1x_2+\lambda_2x_2'),\mu(x_3,x_4)\big)\\&=\mu\big(\lambda_1\mu(x_1,x_2)+\lambda_2\mu(x_1,x_2'),\mu(x_3,x_4)\big)\\&=\lambda_1\mu\big(\mu(x_1,x_2),\mu(x_3,x_4)\big)+\lambda_2\mu\big(\mu(x_1,x_2'),\mu(x_3,x_4)\big)\\&=\lambda_1f(x_1,x_2,x_3,x_4)+\lambda_2f(x_1,x_2',x_3,x_4),\end{aligned}$$

$$\begin{aligned}f(x_1,x_2,\lambda_1x_3+\lambda_2x_3',x_4)&=\mu\big(\mu(x_1,x_2),\mu(\lambda_1x_3+\lambda_2x_3',x_4)\big)\\&=\mu\big(\mu(x_1,x_2),\lambda_1\mu(x_3,x_4)+\lambda_2\mu(x_3',x_4)\big)\\&=\lambda_1\mu\big(\mu(x_1,x_2),\mu(x_3,x_4)\big)+\lambda_2\mu\big(\mu(x_1,x_2),\mu(x_3',x_4)\big)\\&=\lambda_1f(x_1,x_2,x_3,x_4)+\lambda_2f(x_1,x_2,x_3',x_4),\end{aligned}$$

$$\begin{aligned}f(x_1,x_2,x_3,\lambda_1x_4+\lambda_2x_4')&=\mu\big(\mu(x_1,x_2),\mu(x_3,\lambda_1x_4+\lambda_2x_4')\big)\\&=\mu\big(\mu(x_1,x_2),\lambda_1\mu(x_3,x_4)+\lambda_2\mu(x_3,x_4')\big)\\&=\lambda_1\mu\big(\mu(x_1,x_2),\mu(x_3,x_4)\big)+\lambda_2\mu\big(\mu(x_1,x_2),\mu(x_3,x_4')\big)\\&=\lambda_1f(x_1,x_2,x_3,x_4)+\lambda_2f(x_1,x_2,x_3,x_4').\end{aligned}$$

Hence, $f(x_1,x_2,x_3,x_4)=\mu\big(\mu(x_1,x_2),\mu(x_3,x_4)\big)$ satisfies the linearity then (V,f) is 4-algebra.

Similar method is applied for equations $f(x_1,x_2,x_3,x_4)=\mu\big(x_1,\mu(\mu(x_2,x_3),x_4)\big)$, $f(x_1,x_2,x_3,x_4)=\mu(\mu(\mu(x_1,x_2),x_3),x_4)$, $f(x_1,x_2,x_3,x_4)=\mu\Big(x_1,\mu\big(x_2,\mu(x_3,x_4)\big)\Big)$, and $f(x_1,x_2,x_3,x_4)=\mu\big(x_1,\mu(\mu(x_2,x_3),x_4)\big)$. It shows that the above equations also satisfy linearity. From the observation, 4-algebra arises from algebra can be define as follows:

Definition 5: Suppose V is a vector space. Let (V,μ) be an algebra where $\mu: V\times V\to V$ is bilinear. If the operation $f: V\times V\times V\times V\to V$ is linearity and satisfies one of the following

$$f(x_1,x_2,x_3,x_4)=\mu\big(\mu(x_1,x_2),\mu(x_3,x_4)\big), \tag{3}$$

$$f(x_1,x_2,x_3,x_4)=\mu(\mu(\mu(x_1,x_2),x_3),x_4), \tag{4}$$

$$f(x_1,x_2,x_3,x_4)=\mu\Big(x_1,\mu\big(x_2,\mu(x_3,x_4)\big)\Big), \tag{5}$$

$$f(x_1, x_2, x_3, x_4) = \mu\big(\mu\big(x_1, \mu(x_2, x_3)\big), x_4\big), \tag{6}$$

$$f(x_1, x_2, x_3, x_4) = \mu\big(x_1, \mu(\mu(x_2, x_3), x_4)\big), \tag{7}$$

then, (V, f) is called 4-algebra.

By using the similar method for : $V \times V \times V \times V \times V \to V$, 5-algebra which is arising from binary algebra, (V, μ) shown in the following definition.

Definition 6. Suppose V is a vector space. Let (V, μ) be an algebra where $\mu: V \times V \to V$ is bilinear. If the operation $f: V \times V \times V \times V \times V \to V$ is linearity and satisfies one of the following

$$f(x_1, x_2, x_3, x_4, x_5) = \mu(\mu(\mu(\mu(x_1, x_2), x_3), x_4), x_5), \tag{8}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\big(\mu\big(\mu\big(x_1, \mu(x_2, x_3)\big), x_4\big), x_5\big), \tag{9}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\big(\mu\big(x_1, \mu(\mu(x_2, x_3), x_4)\big), x_5\big), \tag{10}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\big(x_1, \mu(\mu(\mu(x_2, x_3), x_4), x_5)\big), \tag{11}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\left(\mu\left(x_1, \mu\big(x_2, \mu(x_3, x_4)\big)\right), x_5\right), \tag{12}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\left(x_1, \mu\big(\mu\big(x_2, \mu(x_3, x_4)\big), x_5\big)\right), \tag{13}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\left(x_1, \mu\big(x_2, \mu(\mu(x_3, x_4), x_5)\big)\right), \tag{14}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\left(x_1, \mu\left(x_2, \mu\big(x_3, \mu(x_4, x_5)\big)\right)\right), \tag{15}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\big(\mu\big(\mu(x_1, x_2), \mu(x_3, x_4)\big), x_5\big), \tag{16}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\big(\mu(x_1, x_2), \mu(\mu(x_3, x_4), x_5)\big), \tag{17}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\left(\mu\big(x_1, \mu(x_2, x_3)\big), \mu(x_4, x_5)\right), \tag{18}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\left(x_1, \mu\big(\mu(x_2, x_3), \mu(x_4, x_5)\big)\right), \tag{19}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\big(\mu(\mu(x_1, x_2), x_3), \mu(x_4, x_5)\big), \tag{20}$$

$$f(x_1, x_2, x_3, x_4, x_5) = \mu\left(\mu(x_1, x_2), \mu\big(x_3, \mu(x_4, x_5)\big)\right). \tag{21}$$

then, (V, f) is called 5-algebra.

Next section gives the classification of 3-algebras and 4-algebras which are arising from algebra by applying the classes of Lie, associative and Leibniz algebras into Definitions 4 and 5.

CLASSIFICATION TWO DIMENSIONAL OF n-ALGEBRAS

This section provides the classification of two dimensional n-Lie, n-associative and n-Leibniz algebras for $n = 3, 4$ that arising from Lie, associative and Leibniz algebras over complex field.

The classification of two dimensional complex 3-Lie, 3-associative and 3-Leibniz algebras by using the equation (1) are presented in Propositions 1, 2 and 3, respectively. The provings are similar and the details of the proving are provided in Proposition 3.

Let $\{e_1, e_2\}$ be a basis for two-dimensional algebra. The equations (1) and (2) can be rewrite in term of e_1, e_2 as follows:

$$[e_i e_j e_k] = [e_i [e_j e_k]], \qquad (1^*)$$

$$[e_i e_j e_k] = [[e_i e_j] e_k], \qquad (2^*)$$

for $i, j, k = 1, 2$.

Proposition 1: For any vector space V. Let $\mu: V \times V \to V$ and $f: V \times V \times V \to V$ be bilinear and trilinear operations, respectively. Then two dimensional complex 3-Lie algebra that arising from Lie algebra satisfies $f(x_1, x_2, x_3) = \mu(x_1, \mu(x_2, x_3))$ is

$$L_3^1: [e_1 e_1 e_2] = e_2, [e_1 e_2 e_1] = -e_2.$$

Proof:
The class of two-dimensional Lie algebra is $L_1^1: [e_1 e_2] = e_2$, $[e_2 e_1] = -e_2$ (refer Theorem 1). By substituting $[e_1 e_2] = e_2$, $[e_2 e_1] = -e_2$ into (1*), the triple multiplications become

$$[e_1 e_1 e_1] = [e_1 e_2 e_2] = [e_2 e_1 e_1] = [e_2 e_1 e_2] = [e_2 e_2 e_1] = [e_2 e_2 e_2] = 0,$$
$$\text{and } [e_1 e_2 e_1] = -e_2, \ [e_1 e_1 e_2] = e_2.$$

Hence, 3-Lie algebra L_3^1: $[e_1 e_1 e_2] = e_2$, $[e_1 e_2 e_1] = -e_2$ is obtained.

Proposition 2: For any vector space V. Suppose $\mu: V \times V \to V$ and $f: V \times V \times V \to V$ are bilinear and trilinear operations, respectively. Then two dimensional complex 3-associative algebras that arising from associative algebras satisfies $f(x_1, x_2, x_3) = \mu\big(x_1, \mu(x_2, x_3)\big)$ is

$$As_3^1: Abelian; \quad As_3^2: e_1 e_1 e_1 = e_1, e_1 e_1 e_2 = e_2; \quad As_3^3: e_1 e_1 e_1 = e_1, e_2 e_1 e_1 = e_2;$$

$$As_3^4: e_1 e_1 e_1 = e_1, e_2 e_2 e_2 = e_2;$$

$$As_3^5: e_1 e_1 e_1 = e_1, e_1 e_1 e_2 = e_2, e_1 e_2 e_1 = e_2, e_2 e_1 e_1 = e_2.$$

Proof:

Referring to Theorem 2, we have five classes of two dimensional associative algebras. By using the similar calculation as in 3-Lie case, we get:

From As_2^1, all triple multiplications are zero. It implies an abelian 3-associative algebra..
As_2^2 gives $e_1e_2e_1 = e_1e_2e_2 = e_2e_1e_2 = e_2e_2e_1 = e_2e_2e_2 = 0$, $e_1e_1e_1 = e_1$ and $e_1e_1e_2 = e_2$.
Then As_3^2: $e_1e_1e_1 = e_1, e_1e_1e_2 = e_2$ is 3-associative algebra.
Associative algebra As_2^3 gives the triple multiplications as $e_1e_1e_1 = e_1$, $e_2e_1e_1 = e_2$ and $e_1e_1e_2 = e_1e_2e_1 = e_1e_2e_2 = e_2e_1e_1 = e_2e_1e_2 = e_2e_2e_1 = e_2e_2e_2 = 0$.
Therefore As_3^3: $e_1e_1e_1 = e_1, e_2e_1e_1 = e_2$ is obtained.
For As_2^4 , we get 3-associative algebra as As_3^4: $e_1e_1e_1 = e_1, e_2e_2e_2 = e_2$ which is the triple multiplications are $e_1e_1e_1 = e_1$, $e_1e_1e_2 = e_1e_2e_1 = e_1e_2e_2 = e_2e_1e_1 = e_2e_1e_2 = e_2e_2e_1 = 0$ and $e_2e_2e_2 = e_2$.
The last As_2^5 gives $e_1e_1e_1 = e_1$, $e_1e_2e_2 = e_2e_1e_2 = e_2e_2e_1 = e_2e_2e_2 = 0$ and $e_1e_1e_2 = e_1e_2e_1 = e_2e_1e_1 = e_2$.
Hence As_3^5: $e_1e_1e_1 = e_1, e_1e_1e_2 = e_2, e_1e_2e_1 = e_2$, $e_2e_1e_1 = e_2$.

Proposition 3: Suppose V is a vector space. Let $\mu: V \times V \to V$ and $f: V \times V \times V \to V$ be a bilinear and a trilinear map, respectively. Then two dimensional complex 3-Leibniz algebras arising from Leibniz algebras satisfies $f(x_1, x_2, x_3) = \mu(x_1, \mu(x_2, x_3))$ is

$$Lb_3^1: Abelian;\ \ Lb_3^2: [e_1, e_1, e_2] = e_2, [e_1, e_2, e_1] = -e_2.$$

Proof:
From Theorem 3, we have Lb_2^1, Lb_2^2 and Lb_2^3.

By applying Lb_2^1 into the equation (1*), we have the following expressions:
$[e_1, e_1, e_1] = [e_1, [e_1, e_1]] = [e_1, e_2] = 0$, since $[e_1, e_1] = e_2$ and $[e_1, e_2] = 0$,
$[e_1, e_i, e_j] = [e_1, [e_i, e_j]] = [e_1, 0] = 0$, since $[e_i, e_j] = 0$, for $i, j = 1, 2$, and $i \neq j$,
$[e_1, e_2, e_2] = [e_1, [e_2, e_2]] = [e_1, 0] = 0$, since $[e_2, e_2] = 0$,
$[e_2, e_1, e_1] = [e_2, [e_1, e_1]] = [e_2, e_2] = 0$, since $[e_1, e_1] = e_2$ and $[e_2, e_2] = 0$,
$[e_2, e_i, e_j] = [e_2, [e_i, e_j]] = [e_2, 0] = 0$, since $[e_i, e_j] = 0, for\ i, j = 1, 2$, and $i \neq j$,
$[e_2, e_2, e_2] = [e_2, [e_2, e_2]] = [e_2, 0] = 0$, since $[e_2, e_2] = 0$.
Thus, we obtain Lb_3^1 as an abelian 3-Leibniz algebra.

For Lb_2^2, we get
$[e_1, e_1, e_1] = [e_1, [e_1, e_1]] = [e_1, 0] = 0$, since $[e_1, e_1] = 0$,
$[e_1, e_1, e_2] = [e_1, [e_1, e_2]] = [e_1, e_2] = e_2$, since $[e_1, e_2] = e_2$,
$[e_1, e_2, e_1] = [e_1, [e_2, e_1]] = [e_1, -e_2] = -[e_1, e_2] = -e_2$, since $[e_2, e_1] = -e_2$ and $[e_1, e_2] = e_2$,
$[e_1, e_2, e_2] = [e_1, [e_2, e_2]] = [e_1, 0] = 0$, since $[e_2, e_2] = 0$,
$[e_2, e_1, e_1] = [e_2, [e_1, e_1]] = [e_2, 0] = 0$, since $[e_1, e_1] = 0$,
$[e_2, e_1, e_2] = [e_2, [e_1, e_2]] = [e_2, e_2] = 0$, since $[e_1, e_2] = e_2$ and $[e_2, e_2] = 0$,
$[e_2, e_2, e_1] = [e_2, [e_2, e_1]] = [e_2, -e_2] = -[e_2, e_2] = 0$, since $[e_2, e_1] = -e_2$ and $[e_2, e_2] = 0$,
$[e_2, e_2, e_2] = [e_2, [e_2, e_2]] = [e_2, 0] = 0$, since $[e_2, e_2] = 0$.
Therefore Lb_3^2: $[e_1, e_1, e_2] = e_2, [e_1, e_2, e_1] = -e_2$ is obtained.

Repeat the same calculation for Lb_2^3. The abelian 3-Leibniz algebra is appeared where all triple multiplications are zero (similar as in Lb_2^1).

Therefore, 3-Leibniz algebras is Lb_3^1: abelian and Lb_3^2: $[e_1, e_1, e_2] = e_2$, $[e_1, e_2, e_1] = -e_2$.

Next, by using equation (2), the classification of two dimensional complex 3-Lie, 3-associative and 3-Leibniz algebras that arising from Theorems 1, 2 and 3 are show in the following propositions.

Proposition 4: For any vector space V. Suppose $\mu: V \times V \to V$ and $f: V \times V \to V$ are bilinear and trilinear operations, respectively. Then two dimensional complex 3-Lie algebra arising from Lie algebra satisfies $f(x_1, x_2, x_3) = \mu(\mu(x_1, x_2), x_3)$ is an abelian.
.
Proof:
Theorem1 shows L_1^1: $[e_1 e_2] = e_2$, $[e_2 e_1] = -e_2$. By substituting it into equation (2*) then an abelian 3-Lie algebra is obtained, where all triple multiplications is zero.

Proposition 5: For any vector space. Let $\mu: V \times V \to V$ and $f: V \times V \times V \to V$ be bilinear and trilinear maps, respectively. Then two dimensional complex 3-associative algebra arising from associative algebra satisfies $f(x_1, x_2, x_3) = \mu(\mu(x_1, x_2), x_3)$ is

$$As_3^1: Abelian; \quad As_3^2: e_1e_1e_1 = e_1, e_1e_1e_2 = e_2; \quad As_3^3: e_1e_1e_1 = e_1, e_2e_1e_1 = e_2;$$

$$As_3^4: e_1e_1e_1 = e_1, e_2e_2e_2 = e_2;$$

$$As_3^5: e_1e_1e_1 = e_1, e_1e_1e_2 = e_2, e_1e_2e_1 = e_2, e_2e_1e_1 = e_2.$$

Proof:
From the list of two dimensional associative algebras in Theorem 2, we applied equation (2*) to get the 3-associative algebras as follows:
From As_2^1, we have $e_ie_je_k = 0$ for all $i, j, k = 1, 2$. Thus, the 3-associative algebra As_3^1 is an abelian.
As_2^2 gives $e_1e_2e_1 = e_1e_2e_2 = e_2e_1e_2 = e_2e_2e_1 = e_2e_2e_2 = 0$, $e_1e_1e_1 = e_1$, $e_1e_1e_2 = e_2$. Then 3-associative algebra is As_3^2: $e_1e_1e_1 = e_1, e_1e_1e_2 = e_2$.
As_2^4 obtains $e_1e_1e_2 = e_1e_2e_1 = e_1e_2e_2 = e_2e_1e_2 = e_2e_2e_1 = e_2e_2e_2 = 0$, $e_1e_1e_1 = e_1$ and $e_2e_1e_1 = e_2$. It implies As_3^3: $e_1e_1e_1 = e_1, e_2e_1e_1 = e_2$ as 3-associative algebra.
As_2^4 gives 3-associative algebra As_3^4: $e_1e_1e_1 = e_1, e_2e_2e_2 = e_2$ where $e_1e_1e_1 = e_1$, $e_1e_1e_2 = e_1e_2e_1 = e_1e_2 = e_2e_1e_1 = e_2e_1e_2 = e_2e_2e_1 = 0$, and $e_2e_2e_2 = e_2$.
Next from As_2^5, we have $e_1e_2e_2 = e_2e_1e_2 = e_2e_2e_1 = e_2e_2e_2 = 0$, $e_1e_1e_1 = e_1$ and $e_1e_1e_2 = e_1e_2e_1 = e_2e_1e_1 = e_2$.
Therefore As_3^5: $e_1e_1e_1 = e_1, e_1e_1e_2 = e_2, e_1e_2e_1 = e_2$, $e_2e_1e_1 = e_2$ is obtained.

Proposition 6: For any vector space V. Let $\mu: V \times V \to V$ and $f: V \times V \times V \to V$ be bilinear and trilinear operations, respectively. Then two dimensional complex 3-Leibniz algebra arising from Leibniz algebra satisfies $f(x_1, x_2, x_3) = \mu(\mu(x_1, x_2), x_3)$ is

$$Lb_3^1: Abelian,; \quad Lb_3^2: [e_1, e_2, e_1] = -e_2, [e_2, e_1, e_1] = e_2;$$

$$Lb_3^3: [e_1, e_2, e_2] = e_1, [e_2, e_2, e_2] = e_1.$$

Proof:

From Theorem3 and equation (2*), we have Lb_2^1 gives $[e_i, e_j, e_k] = 0$ for all $i, j, k = 1, 2$. This implies algebra Lb_3^1 as an abelian. Lb_2^2 gives $[e_1, e_2, e_1] = -e_2, [e_2, e_1, e_1] = e_2$ and other triple multiplications are zero. Thus Lb_3^2: $[e_1, e_2, e_1] = -e_2, [e_2, e_1, e_1] = e_2$ is obtained. Lastly, Lb_2^3 gives the algebra Lb_3^3: $[e_1, e_2, e_2] = e_1, [e_2, e_2, e_2] = e_1$.

Propositions 7, 8 and 9 show the classification of two dimensional 4-Lie, 4-associative and 4-Leibniz algebras which are arising from the classification of two dimensional Leibniz, associative and Lie algebras by using equations (3), (4) and (5), respectively. Since $\{e_1, e_2\}$ is a basis for two dimensional algebra, these equations can be express as

$$[e_i e_j e_k e_l] = \left[[e_i e_j][e_k e_l]\right], \qquad (3^*)$$

$$[e_i e_j e_k e_l] = [[[e_i e_j] e_k] e_l], \qquad (4^*)$$

$$[e_i e_j e_k e_l] = [e_i [e_j [e_k e_l]]], \qquad (5^*)$$

for $i, j, k, l = 1, 2$.

Proposition 7: For any vector space V. Suppose $f: V \times V \times V \times V \to V$ is defined by $f(x_1, x_2, x_3, x_4) = \mu(\mu(x_1, x_2), \mu(x_3, x_4))$, where $\mu: V \times V \to V$. Then the isomorphism classes of two dimensional complex

1. 4-Lie algebra arising from Leibniz algebra is an abelian 4-Lie algebra.
2. 4-associative algebra arising from associative algebra is

$$As_4^1: Abelian; \quad As_4^2: e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2;$$

$$As_4^3: e_1e_1e_1e_1 = e_1, e_2e_1e_1e_1 = e_2; \quad As_4^4: e_1e_1e_1e_1 = e_1, e_2e_2e_2e_2 = e_2;$$

$$As_4^5: e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2, e_1e_1e_2e_1 = e_2, e_1e_2e_1e_1 = e_2, e_2e_1e_1e_1 = e_2.$$

3. 4-Leibniz algebra arising from Lie algebra is an abelian 4-Leibniz algebra.

Proof:

1. From Theorem 1, there is only one class of Lie algebras in dimension two which is L_2^1: $[e_1 e_2] = e_2$. By substituting it into equation (3*) we will find an abelian 4-Lie algebras since all multiplications are zero.

2. Theorem 2 gives five classes of associative algebras denoted as As_2^1, As_2^2, As_2^3, As_2^4 and As_2^5. The following shows the substituting these classes into equation (3*):
   Associative algebra As_2^1 gives an abelian 4-associative algebra where $e_ie_je_ke_l = 0$ for all $i, j, k, l = 1, 2$.
   For As_2^2, we have $e_2e_2e_2e_1 = e_2e_2e_2e_2 = e_1e_1e_1e_1 = e_1$, $e_1e_1e_1e_2 = e_2$ and other multiplications equal to zero. Hence, 4-associative algebra is $As_4^2: e_1e_1e_1e_1 = e_1, e_2e_2e_2e_1 = e_1, e_2e_2e_2e_2 = e_1, e_1e_1e_1e_2 = e_2$.

Now for As_2^3, we obtain $e_1e_1e_1e_1 = e_1, e_2e_1e_1e_1 = e_2$ and other multiplications equal to zero. Thus, 4-associative algebra is $As_4^3: e_1e_1e_1e_1 = e_1, e_2e_1e_1e_1 = e_2$.
Next we get 4-associative algebra is $As_4^4: e_1e_1e_1e_1 = e_1, e_2e_2e_2e_2 = e_2$ where $e_1e_1e_1e_1 = e_1,\ e_2e_2e_2e_2 = e_2$ and other multiplications equal to zero are the results of As_2^4.
Finally As_2^5 gives $e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2, e_1e_1e_2e_1 = e_2,\ e_1e_2e_1e_1 = e_2$, $e_2e_1e_1e_1 = e_2$ and other multiplications equal to zero. Therefore, $As_4^5: e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2, e_1e_1e_2e_1 = e_2, e_1e_2e_1e_1 = e_2, e_2e_1e_1e_1 = e_2$ is obtained.

3. Theorem 3 shows three classes of Leibniz algebras in dimension two, which are Lb_2^1, Lb_2^2 and Lb_2^3. By substituting them one by one into equation (3*), give $[e_i, e_j, e_k, e_l] = 0$ for all $i, j, k, l = 1, 2$. Thus, we get an abelian 4-Leibniz algebra.

Proposition 8: For any vector space V. Suppose $f: V \times V \times V \times V \to V$ is defined by $f(x_1, x_2, x_3, x_4) = \mu\left(x_1, \mu\left(x_2, \mu(x_3, x_4)\right)\right)$, where $\mu: V \times V \to V$. Then the isomorphism classes of two dimensional complex

1. 4-Lie algebra arising from Lie algebra is $L_4^1: [e_1e_1e_1e_2] = e_2$.

2. 4-associative algebra arising from associative algebra is

$$As_4^1: Abelian; \quad As_4^2: e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2;$$

$$As_4^3: e_1e_1e_1e_1 = e_1, e_2e_1e_1e_1 = e_2; \ As_4^4: e_1e_1e_1e_1 = e_1, e_2e_2e_2e_2 = e_2;$$

$$As_4^5: e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2, e_1e_1e_2e_1 = e_2, e_1e_2e_1e_1 = e_2, e_2e_1e_1e_1 = e_2.$$

3. 4-Leibniz algebra arising from Leibniz algebra is

$$Lb_4^1: Abelian; \quad Lb_4^2: [e_1, e_1, e_1, e_2] = e_2, [e_1, e_1, e_2, e_1] = -e_2.$$

Proof:

1. The class of Lie algebra (in Theorem 1) is L_2^1. Substitute it into equation (4*) and we find 4-Lie algebras is $L_4^1: [e_1e_1e_1e_2] = e_2,\ [e_1e_1e_2e_1] = -e_2$ since all multiplications equal zero except $[e_1e_1e_1e_2] = e_2$ and $[e_1e_1e_2e_1] = -e_2$.

2. By substituting five classes of associative algebra from Theorem 2 into equation (4*), the classes of 4-associative algebras obtained as follows:
From As_2^1, $e_ie_je_ke_l = 0$ for all $i, j, k, l = 1, 2$. Then, As_4^1 is an abelian.
For As_2^2, $e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2$ and other multiplications equal to zero, implies $As_4^2: e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2$ is 4-associatice algebra.
Consider As_2^3, $e_1e_1e_1e_1 = e_1, e_2e_1e_1e_1 = e_2$ and other multiplications equal to zero. It is obtained $As_4^3: e_1e_1e_1e_1 = e_1, e_2e_1e_1e_1 = e_2$.
As_2^4 shows that $e_1e_1e_1e_1 = e_1, e_2e_2e_2e_2 = e_2$ and other multiplications equal to zero, therefore 4-associative algebra is $As_4^4: e_1e_1e_1e_1 = e_1, e_2e_2e_2e_2 = e_2$.
Since As_2^5 gives $e_1e_1e_1e_1 = e_1$, $e_1e_1e_1e_2 = e_1e_1e_2e_1 = e_1e_2e_1e_1 = e_2e_1e_1e_1 = e_2$ and other multiplication equal to zero, it implies
$$As_4^5: e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2, e_1e_1e_2e_1 = e_2, e_1e_2e_1e_1 = e_2, e_2e_1e_1e_1 = e_2.$$

3. From Theorem 3, we have Lb_2^1, Lb_2^2 and Lb_2^3. By substituting them into equation (4*), we find Leibniz 4-algebras as:
   For Lb_2^1 and Lb_2^3, $[e_i, e_j, e_k, e_l] = 0$ for all $i, j, k, l = 1, 2$ are obtained. Therefore, Lb_4^1 is an abelian. As for Lb_2^2, $[e_1, e_1, e_1, e_2] = e_2, [e_1, e_1, e_2, e_1] = e_2$ and other multiplications equal to zero gives Lb_4^2: $[e_1, e_1, e_1, e_2] = e_2, [e_1, e_1, e_2, e_1] = -e_2$.

Proposition 9: For any vector space V. Suppose $f: V \times V \times V \times V \to V$ is defined by $f(x_1, x_2, x_3, x_4) = \mu(\mu(x_1, \mu(x_2, x_3)), x_4)$, where $\mu: V \times V \to V$. Then the isomorphism classes of two dimensional complex

1. 4-Lie algebra arising from Lie algebra is an abelian.

2. 4-associative algebra arising from associative algebra is

$$As_4^1: Abelian;\ As_4^2: e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2;$$

$$As_4^3: e_1e_1e_1e_1 = e_1, e_2e_1e_1e_1 = e_2;\ As_4^4: e_1e_1e_1e_1 = e_1, e_2e_2e_2e_2 = e_2;$$

$$As_4^5: e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_1e_1e_2e_1 = e_1e_2e_1e_1 = e_2e_1e_1e_1 = e_2.$$

3. 4-Leibniz algebra arising from Leibniz algebras is

$$Lb_4^1: Abelian;\ Lb_4^2 = [e_1, e_1, e_2, e_1] = -e_2, [e_1, e_2, e_1, e_1] = e_2.$$

Proof:

1. The class of Lie algebra from Theorem 1 is L_2^1. Substitute it into equation (5*) gives $[e_ie_je_ke_l] = 0$ for all $i, j, k, l = 1, 2$. Therefore, 4-Lie algebra in dimension two, L_4^1 is an abelian.

2. By substituting five classes of associative algebra from Theorem 2 into equation (5*), 4-associative algebras is obtained as follows:
   Since As_2^1 gives $e_ie_je_ke_l = 0$ for all $i, j, k, l = 1, 2$, then As_4^1 is abelian.
   Given $e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2$ and other multiplications are zero from As_2^2, then As_4^2: $e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2$.
   As_2^3 gives $e_1e_1e_1e_1 = e_1, e_2e_1e_1e_1 = e_2$ and other multiplications are zero, then As_4^3: $e_1e_1e_1e_1 = e_1, e_2e_1e_1e_1 = e_2$ is obtained.
   Next, we have $e_1e_1e_1e_1 = e_1, e_2e_2e_2e_2 = e_2$ and other multiplications are zero from As_2^4. Thus 4-associative algebra is As_4^4: $e_1e_1e_1e_1 = e_1, e_2e_2e_2e_2 = e_2$.
   Now consider As_2^5, $e_1e_1e_1e_1 = e_1$, $e_1e_1e_1e_1 = e_1$, $e_1e_1e_1e_2 = e_2$, $e_1e_1e_2e_1 = e_2$, $e_1e_2e_1e_1 = e_2$, $e_2e_1e_1e_1 = e_2$ and other multiplications are zero implies 4-associative algebras as
   $$As_4^5: e_1e_1e_1e_1 = e_1, e_1e_1e_1e_2 = e_2, e_1e_1e_2e_1 = e_2, e_1e_2e_1e_1 = e_2, e_2e_1e_1e_1 = e_2.$$

3. Theorem 3 has three classes, Lb_2^1, Lb_2^2, Lb_2^3. Substitute them into equation (5*) will obtain Leibniz 4-algebras as follows:

   From Lb_2^1 and Lb_2^3, $[e_i, e_j, e_k, e_l] = 0$ for all $i, j, k, l = 1, 2$, then Lb_4^1: Abelian.

For Lb_2^2, we find $[e_1, e_1, e_2, e_1] = -e_2, [e_1, e_2, e_1, e_1] = e_2$ and other multiplications are zero, therefore $Lb_4^2 = [e_1, e_1, e_2, e_1] = -e_2, [e_1, e_2, e_1, e_1] = e_2$.

CONCLUSION

This work investigates the relation between n-algebra and binary algebra or algebra. For $n = 3, 4, 5$, Definitions 4 until 6 show that n-algebra can be arising from binary algebra. The classification of n-algebra for $n = 3, 4$, in cases of n-Lie, n-associative and n-Leibniz algebras are shown in Propositions 1 until 9.

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