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Existence of a Unique Solution of an Infinite Three-Coupled System Model of Ordinary Differential Equations in Hilbert Space

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ABSTRACT

Numerous practical problems are often formulated as control problems modeled by partial differential equations (PDE). The spectral decomposition method often transforms such problems into an equivalent control problem modeled by an infinite system of ordinary differential equations (ODE). We examine a solution to an infinite system of three-coupled ODE in the Hilbert space, ℓ_2. The existence and uniqueness of the solution to the given infinite system in the Hilbert space ℓ_2 are proved as our main research findings. Our findings imply that control and differential games modeled by this infinite system of three-coupled ODE can now be investigated.

Keywords: existence and uniqueness, Hilbert space, infinite system, ordinary differential equations, three-coupled

INTRODUCTION

Control problems in defence, engineering, industry, etc are often modelled using hyperbolic and parabolic partial differential equations (PDE). Several scholars have employed diverse techniques to tackle control problems for hyperbolic and parabolic PDE (see, for instance, Avdonin and Ivanov (1995), Butkovskiy (1969), Chernous'ko (1992), Ibragimov (2003), Satimov and Tukhtasinov (2006)). The spectral decomposition technique can convert some of these control problems modeled using PDE to control problems modeled using an infinite system of ordinary differential equations (ODE) (see Axelband (1967), Russell (1978), Tukhtasinov (1995), Fursikov (2000), Tukhtasinov and Mamatov (2008)). For instance, Satimov and Tukhtasinov (2006) considered the parabolic equation

$$z_t = Az - u + v, \qquad z\,\big|_{t=0} = z_0(x), \qquad z\,\big|_{S_T} = 0$$

such that

$$Az = \sum_{l,m=1}^{n} \frac{\partial}{\partial x_l}\left[a_{lm}(x)\,\frac{\partial z}{\partial x_m}\right], \qquad a_{lm}(x) = a_{ml}(x),$$

satisfies the inequality

$$\sum_{l,m=1}^{n} a_{lm}(x)\chi_l\chi_m \geq \gamma \sum_{l=1}^{n} \chi_l^2$$

A unique solution $z = z(x,t)$ to the resulting game system was shown to be of the form

$$z(x,t) = \sum_{l=1}^{\infty} z_l(t)\,\psi_l(x)$$

which also happens to be the solution for the infinite system of ODE:

$$\dot{z}_l = \lambda_l z_l - u_l(t) + v_l(t), \quad z_l(0) = z_{l0}, \quad l = 1, 2, \ldots$$

where the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_l, \ldots$ of A satisfy

$$\cdots \leq \lambda_l \leq \cdots \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 < 0 \quad \text{and} \quad \lambda_l \to -\infty \quad \text{as } l \to \infty$$

Satimov and Tukhtasinov (2006) highlighted the significance of examining an infinite system of ODE due to its strong link to hyperbolic or parabolic PDE. Nevertheless, as demonstrated in Alias et al (2017), Ibragimov et al. (2008) and Ibragimov et al. (2022a), the infinite system can be examined without recourse to the PDE problem.

Recall that the vector space consisting of square-summable real number sequences

$$\ell_2 = \left\{\chi = \left(\chi_1, \chi_2, \chi_3, \ldots\right) \sum_{l=1}^{\infty} \chi_l^2 < \infty \right\}$$

is a Hilbert space endowed with inner product and vector norm given respectively by

$$(\chi, \xi) = \sum_{l=1}^{\infty} \chi_l \xi_l, \quad \|\chi\| = \sqrt{(\chi, \chi)} \qquad \text{for} \qquad \chi, \xi \in \ell_2.$$

The study of unique solutions of infinite systems of coupled ODE existing in ℓ_2-space has attracted several research interests of recent because of the necessity of establishing the uniqueness of solutions in the ℓ_2-space before control and pursuit-evasion differential games can be meaningfully solved on such infinite systems in the space.

Ibragimov (2004) proved that a unique solution for the infinite system

$$\dot{z}_l = -\lambda_l z_l + \varpi_l(t), \qquad z_l(0) = z_{l0}$$

exists in the ℓ_2-space, where $\lambda_l \geq 0, l = 1, 2, \ldots$ and $z_0 = (z_{10}, z_{20}, \ldots) \in \ell_2$. In Ibragimov et al. (2008), a unique solution for the infinite system of two-coupled ODE

$$\begin{aligned} \dot{x}_l &= -\alpha_l x_l - \beta_l y_l + \varpi_{l1}(t), \quad x_l(0) = x_{l0} \\ \dot{y}_l &= \beta_l x_l - \alpha_l y_l + \varpi_{l2}(t), \quad y_l(0) = y_{l0} \end{aligned} \tag{1}$$

was shown to exist in the Hilbert space ℓ_2, where $\alpha_l \geq 0, \beta_l \in \mathbb{R}, l = 1, 2, \ldots$, where

$$x_0 = (x_{10}, x_{20}, \ldots) \in \ell_2 \quad \text{and} \quad y_0 = (y_{10}, y_{20}, \ldots) \in \ell_2.$$

Ibragimov and Kuchkarova (2021) proved that a unique solution of (1) exists in ℓ_2-space for the case $\alpha_l \leq 0, \beta_l \in \mathbb{R}$.

In Madhavan et al. (2024a), a unique solution for the following infinite system of three-coupled ODEs:

$$\begin{aligned} \dot{x}_l &= -\mu_l x_l + \varpi_{l1}(t), & x_l(0) &= x_{l0} \\ y_l &= -\lambda_l y_l - \gamma_l z_l + \varpi_{l2}(t), & y_l(0) &= y_{l0} \\ z_l &= \gamma_l y_l - \lambda_l z_l + \varpi_{l3}(t), & z_l(0) &= z_{l0} \end{aligned} \tag{2}$$

was shown to exist in the ℓ_2-space, where $\alpha_l, \beta_l \geq 0$, $\gamma_l \in \mathbb{R}$, $\varpi_{l1}, \varpi_{l2}, \varpi_{l3} \in \mathbb{R}$, $l = 1, 2, \ldots$, $x_0, y_0, z_0 \in \ell_2$ and

$$x_0 = (x_{10}, x_{20}, \ldots) \in \ell_2, \quad y_0 = (y_{10}, y_{20}, \ldots) \in \ell_2 \quad \text{and} \quad z_0 = (z_{10}, z_{20}, \ldots) \in \ell_2.$$

In Ruziboev et al. (2023), it was proved that if $\{A_l\}$ is a family of uniformly normalizable matrices (with dimensions $d_l \times d_l$, $2 \leq d_l \leq d$, where d is a fixed integer), such that the eigenvalues of the A_l's $l = 1, 2, \ldots$, have negative real parts, then for any admissible control ϖ, a unique solution to the system

$$\dot{X}_l = A_l X_l + \varpi_l, \quad X_l(0) = X_{l0} \in \mathbb{R}^{d_l}, \quad l = 1, 2, \ldots, \tag{3}$$

exists exists in ℓ_2. This result of Ruziboev et al. (2023) extended the result of Madhavan et al. (2024a) since the coefficient matrix in (2) has eigenvalues with negative real parts.
The study of unique solutions of infinite systems of other coupled ODE existing in ℓ_2-space have also been recently conducted (see for instance, Ibragimov et al. (2022c), Qushakov (2022), Alias et al. (2023)).

Motivated by recent research trends discussed above, we will study for the first time, the existence of unique solution of the infinite system of three-coupled ODE

$$\begin{aligned} \dot{x}_l &= \mu_l x_l + \varpi_{l1}(t), & x_l(0) &= x_{l0} \\ y_l &= \lambda_l y_l - \gamma_l z_l + \varpi_{l2}(t), & y_l(0) &= y_{l0} \\ z_l &= \gamma_l y_l + \lambda_l z_l + \varpi_{l3}(t), & z_l(0) &= z_{l0} \end{aligned} \tag{4}$$

in the Hilbert space ℓ_2, where $l = 1, 2, \ldots$, $\mu_l, \lambda_l \geq 0$, $\gamma_l \in \mathbb{R}$, $0 \leq \mu_l \leq \mu$, $0 \leq \lambda_l \leq \lambda$, $\varpi_{l1}, \varpi_{l2}, \varpi_{l3} \in \mathbb{R}$, $t \in [0, T]$ and

$$x_0 = (x_{10}, x_{20}, \ldots) \in \ell_2, \quad y_0 = (y_{10}, y_{20}, \ldots) \in \ell_2 \quad \text{and} \quad z_0 = (z_{10}, z_{20}, \ldots) \in \ell_2.$$

It should be noted that our system (4) is distinct from (2) and that of (3) since the coefficient matrix of our system (4) has eigenvalues with positive real parts.

The main aim of our current study is to establish that a unique solution of (4) exists in the ℓ_2-space and continuous on [0, T].

SOME PRELIMINARIES

We set

$$\chi_l(t) = \left(x_l(t), y_l(t), z_l(t)\right), \quad |\chi_l(t)| = \sqrt{(x_l^2(t) + y_l^2(t) + z_l^2(t))}$$

$$\chi(t) = (\chi_1, \chi_2, \ldots) = (x_1(t), y_1(t), z_1(t), x_2(t), y_2(t), z_2(t), \ldots)$$

$$\chi_0 = (\chi_{10}, \chi_{20}, \ldots) = (x_{10}, y_{10}, z_{10}, x_{20}, y_{20}, z_{20}, \ldots)$$

$$\left\|\chi(t)\right\| = \sqrt{\sum_{l=1}^{\infty}(x_l^2(t) + y_l^2(t) + z_l^2(t))},$$

$$\left\|\chi_0(t)\right\| = \sqrt{\sum_{l=1}^{\infty}(x_{l0}^2(t) + y_{l0}^2(t) + z_{l0}^2(t))}$$

Definition 1: We represent the class of functions $\varpi(t) = (\varpi_1(t), \varpi_2(t), \ldots)$, $\varpi : [0,\ T] \mapsto \ell_2$, with measurable coordinates $\varpi_l(t) = (\varpi_{l1}(t), \varpi_{l2}(t), \varpi_{l3}(t))$, $t \in [0,\ T]$, $l = 1, 2, \ldots$, which satisfies the inequality

$$\sum_{l=1}^{\infty} \int_0^T \left(\sum_{m=1}^{3} \varpi_{lm}^2(s) \right) ds \le \rho_0^2$$

by $S(\rho_0)$, where ρ_0 is a positive real number.

Definition 2: Let $\varpi(\cdot) \in S(\rho_0)$. A function $\chi(t) = (\chi_1(t),\ \chi_2(t), \ldots)$, with coordinates $\chi_l(t)$ continuous and satisfying the initial conditions $\chi_l(0) = \chi_{l0}$, $l = 1, 2, \ldots$, is called a solution of (4) if $\chi_l(t)$ satisfies (4) almost everywhere on the interval $[0,\ T]$ and is differentiable almost everywhere on $[0,\ T]$.

Define the fundamental matrix

$$\Phi_l(t) = \begin{pmatrix} e^{\mu_l t} & 0 & 0 \\ 0 & e^{\lambda_l t}\cos\gamma_l t & -e^{\lambda_l t}\sin\gamma_l t \\ 0 & e^{\lambda_l t}\sin\gamma_l t & e^{\lambda_l t}\cos\gamma_l t \end{pmatrix}, \qquad l = 1,\ 2,\ \ldots.$$

Property 1: It is readily verifiable that the fundamental matrix $\Phi_l(t)$ satisfies these properties:

(a) $\Phi_l^{-1}(t) = \Phi_l(-t)$ and $\Phi_l(0) = E_3$, where $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) $\Phi_l(t+h) = \Phi_l(t)\,\Phi_l(h) = \Phi_l(h)\,\Phi_l(t)$

(c) $\left|\Phi_l(t)\chi_l\right| = \left|\Phi_l^T(t)\chi_l\right| \le e^{\beta_l t}\left|\chi_l\right|$, where $\beta_l = \max\{\mu_l,\ \lambda_l\}$, $\beta = \max_l\{\beta_l\}$, $\chi_l \in \mathbb{R}^3$.

(d) $\left\|\Phi_1(t) - E_3\right\| \le \psi(t)$ where $\psi(t) = e^{\mu t} + 2e^{\lambda t} + 2$.

Here, $\Phi_l^T(t)$ denotes the transpose of matrix $\Phi_l(t)$ and $\left\|\Phi\right\| = \max_{|x|=1}\left|\Phi x\right|$, $x = (x_1,\ x_2,\ x_3)$ is such that $x_1^2 + x_2^2 + x_3^2 = 1$.

We can establish the last inequality of Property 1(d) as follows:

$$\left\|\Phi_1(t) - E_3\right\| = \max_{|x|=1}\left|\left(\Phi_1(t) - E_3\right) x\right|$$
$$= \max_{|x|=1}\left|\left[\begin{pmatrix} e^{\mu_1 t} & 0 & 0 \\ 0 & e^{\lambda_1 t}\cos\gamma_1 t & -e^{\lambda_1 t}\sin\gamma_1 t \\ 0 & e^{\lambda_1 t}\sin\gamma_1 t & e^{\lambda_1 t}\cos\gamma_1 t \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right|$$
$$= \max_{|x|=1}\left|\begin{pmatrix} e^{\mu_1 t} - 1 & 0 & 0 \\ 0 & e^{\lambda_1 t}\cos\gamma_1 t - 1 & -e^{\lambda_1 t}\sin\gamma_1 t \\ 0 & e^{\lambda_1 t}\sin\gamma_1 t & e^{\lambda_1 t}\cos\gamma_1 t - 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right|$$
$$= \max_{|x|=1}\left|\begin{pmatrix} \left(e^{\mu_1 t} - 1\right)x_1 \\ \left(e^{\lambda_1 t}\cos\gamma_1 t - 1\right)x_2 - \left(e^{\lambda_1 t}\sin\gamma_1 t\right)x_3 \\ \left(e^{\lambda_1 t}\sin\gamma_1 t\right)x_2 + \left(e^{\lambda_1 t}\cos\gamma_1 t - 1\right)x_3 \end{pmatrix}\right|$$

Simplifying further, we have:

$$\left\|\Phi_1(t) - E_3\right\| \le \max_{|x|=1}\left(\left|\begin{matrix} \left(e^{\mu_1 t} - 1\right)x_1 \\ 0 \\ 0 \end{matrix}\right| + \left|\begin{matrix} 0 \\ \left(e^{\lambda_1 t}\cos\gamma_1 t - 1\right)x_2 \\ \left(e^{\lambda_1 t}\cos\gamma_1 t - 1\right)x_3 \end{matrix}\right| + \left|\begin{matrix} 0 \\ -\left(e^{\lambda_1 t}\sin\gamma_1 t\right)x_3 \\ \left(e^{\lambda_1 t}\sin\gamma_1 t\right)x_2 \end{matrix}\right|\right)$$
$$= \max_{|x|=1}\left[\left|e^{\mu_1 t} - 1\right|\left|x_1\right| + \left|e^{\lambda_1 t}\cos\gamma_1 t - 1\right|\sqrt{x_2^2 + x_3^2} + \left|e^{\lambda_1 t}\sin\gamma_1 t\right|\sqrt{x_2^2 + x_3^2}\right]$$
$$\le \left|e^{\mu_1 t} - 1\right| + \left|e^{\lambda_1 t}\cos\gamma_1 t - 1\right| + \left|e^{\lambda_1 t}\sin\gamma_1 t\right| \le e^{\mu_1 t} + 1 + e^{\lambda_1 t} + 1 + e^{\lambda_1 t}$$
$$= e^{\mu_1 t} + 2e^{\lambda_1 t} + 2 = \psi(t)$$

The properties of the fundamental matrix $\Phi_l(t)$ enumerated above, are crucial in proving the primary result of this research.

RESULTS

Let $C(0,T;\ell_2)$ be the space consisting of continuous functions $\chi(t) \in \ell_2$, $t \in [0,T]$. We present the two lemmas on which the proof of our main theorem is hinged.

The main result in this article is contained in the following theorem.
Lemma 1: For any given $T > 0$, a unique solution to the infinite system of three-coupled ODE given in (4) exists in the ℓ_2-space, where $\varpi(\cdot) \in S(\rho_0)$, $0 \le \mu_l \le \mu$ and $0 \le \lambda_l \le \lambda$, $l = 1, 2, \ldots$.

Proof: Using the variation of parameters technique, it can be established that each system of three-coupled ODE in the infinite system (4) has a unique solution

$$\chi_l(t) = \Phi_l(t)\chi_{l0} + \Phi_l(t)\int_0^t \Phi_l^{-1}(s)\,\varpi_l(s)\,ds$$

or alternatively,

$$\chi_l(t) = \Phi_l(t)\chi_{l0} + \int_0^t \Phi_l(t-s)\,\varpi_l(s)\,ds, \tag{5}$$

where $\chi_{l0} = (x_{l0}, y_{l0}, z_{l0})$ and $\varpi_l = (\varpi_{l1}, \varpi_{l2}, \varpi_{l3})$. Thus, there cannot be more than one solution of (4).

Next, we show that $\chi(\cdot) = \left(\chi_1(\cdot), \chi_2(\cdot), \ldots\right) \in C(0, T; \ell_2)$. To accomplish this, we first establish that $\chi(t) = \left(\chi_1(t), \chi_2(t), \ldots\right) \in \ell_2$ and then show the continuity of $\chi(t)$ in the norm of ℓ_2-space for each $t \in [0, T]$.

First, for all $t \in [0, T]$, we establish that $\chi(t) = \left(\chi_1(t), \chi_2(t), \ldots\right) \in \ell_2$. Indeed, from (5), we deduce that

$$|\chi_l(t)|^2 \leq 2\left[\left|\Phi_l(t)\chi_{l0}\right|^2 + \left(\int_0^t \left|\Phi_l(t-s)\,\varpi_l(s)\right| ds\right)^2\right]. \tag{6}$$

From Property 1(c), we have that

$$\left|\Phi_l(t-s)\,\varpi_l(s)\right| \leq e^{\beta_l(t-s)}\left|\varpi_l(s)\right| \leq e^{\beta(t-s)}\left|\varpi_l(s)\right| \quad \text{for } 0 \leq s \leq t. \tag{7}$$

Using equations (6) and (7), we find that

$$\begin{aligned} |\chi_l(t)|^2 &\leq 2\left[\left|\Phi_l(t)\chi_{l0}\right|^2 + \left(\int_0^t \left|\Phi_l(t-s)\,\varpi_l(s)\right| ds\right)^2\right] \\ &\leq 2\left[(e^{\beta t})^2\left|\chi_{l0}\right|^2 + \left(\int_0^t e^{\beta(t-s)t}\left|\varpi_l(s)\right| ds\right)^2\right] \end{aligned} \tag{8}$$

The last integral in (8) can be estimated using the Cauchy-Schwarz inequality as follows:

$$\begin{aligned} \left(\int_0^t e^{\beta(t-s)t}\cdot\left|\varpi_l(s)\right| ds\right)^2 &\leq \int_0^t e^{2\beta(t-s)t}\varpi_l(s)\,ds \int_0^t \left|\varpi_l(s)\right|^2 ds \\ &= \varphi(t)\int_0^t \left|\varpi_l(s)\right|^2 ds \end{aligned} \tag{9}$$

where $\varphi(t) = \dfrac{e^{2\beta t} - 1}{2\beta}$. Note also that $\varphi(t) \leq \varphi(T)$ for $0 \leq t \leq T$. Hence, using equation (9) in equation (8), we obtain that

$$|\chi_l(t)|^2 \leq 2\left[e^{2\beta t}\left|\chi_{l0}\right|^2 + \varphi(t)\int_0^t \left|\varpi_l(s)\right|^2 ds\right],$$

and because $t \in [0, T]$, we find that

$$|\chi_l(t)|^2 \leq 2\left[e^{2\beta T}|\chi_{l0}|^2 + \varphi(T)\int_0^T |\varpi_l(s)|^2\, ds\right]. \tag{10}$$

If we sum equation (10) over all positive indices l, we get

$$\begin{aligned}\sum_{l=1}^{\infty}|\chi_l(t)|^2 &\leq 2\left[e^{2\beta T}\sum_{l=1}^{\infty}|\chi_{l0}|^2 + \varphi(T)\sum_{l=1}^{\infty}\int_0^T |\varpi_l(s)|^2\, ds\right] \\ &\leq 2\left[e^{2\beta T}\|\chi_{l0}\|^2 + \varphi(T)\rho_0^2\right]\end{aligned} \tag{11}$$

Hence, we deduce that $\chi(t) \in \ell_2$ for all $t \in [0, T]$. The lemma is thus proved.

Lemma 2: For any given $T > 0$, a unique solution to the infinite system of three-coupled ODE given in (4) is continuous for all $t \in [0, T]$ in the Hilbert space ℓ_2., where $\varpi(\cdot) \in S(\rho_0)$, $0 \leq \mu_l \leq \mu$ and $0 \leq \lambda_l \leq \lambda$, $l = 1, 2, \ldots$.

Proof:
Here, we establish the continuity of the function $\chi(t)$, $t \in [0, T]$ in the Hilbert space ℓ_2. To accomplish this, we show that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|\chi(t+\kappa) - \chi(t)\| < \varepsilon$ whenever $|\kappa| < \delta$.
In doing this, we look at two cases: $\kappa \geq 0$ and $\kappa < 0$.

For the first case, let $\kappa \geq 0$ be given. Using equation (5), we find that

$$\begin{aligned}\chi_l(t+\kappa) - \chi_l(t) &= \Phi_l(t+\kappa)\chi_{l0} + \int_0^{t+\kappa}\Phi_l(t+\kappa-s)\,\varpi_l(s)\,ds - \left[\Phi_l(t)\chi_{l0} + \int_0^t \Phi_l(t-s)\,\varpi_l(s)\,ds\right] \\ &= \Phi_l(t+\kappa)\chi_{l0} - \Phi_l(t)\chi_{l0} + \int_0^t \Phi_l(t+\kappa-s)\,\varpi_l(s)\,ds + \int_t^{t+\kappa}\Phi_l(t+\kappa-s)\,\varpi_l(s)\,ds - \int_0^t \Phi_l(t-s)\,\varpi_l(s)\,ds \\ &= \left[\Phi_l(t+\kappa) - \Phi_l(t)\right]\chi_{l0} + \int_0^t\left[\Phi_l(t+\kappa-s) - \Phi_l(t-s)\right]\varpi_l(s)\,ds + \int_t^{t+\kappa}\Phi_l(t+\kappa-s)\,\varpi_l(s)\,ds\end{aligned}$$

Using Property 1(b) on the last equation, we obtain that

$$\chi_l(t+\kappa) - \chi_l(t) = \left[\Phi_l(\kappa) - E_3\right]\Phi_l(t)\chi_{l0} + \int_0^t\left[\Phi_l(\kappa) - E_3\right]\Phi_l(t-s)\,\varpi_l(s)\,ds + \int_t^{t+\kappa}\Phi_l(t+\kappa-s)\,\varpi_l(s)\,ds \tag{12}$$

Taking the absolute value of both sides of the equation (12) and using the triangle inequality yields

$$|\chi_l(t+\kappa) - \chi_l(t)| \leq \left|\left[\Phi_l(\kappa) - E_3\right]\Phi_l(t)\chi_{l0}\right| + \left|\int_0^t\left[\Phi_l(\kappa) - E_3\right]\Phi_l(t-s)\,\varpi_l(s)\,ds\right| + \left|\int_t^{t+\kappa}\Phi_l(t+\kappa-s)\,\varpi_l(s)\,ds\right| \tag{13}$$

Squaring both sides of (13), and then applying the inequality $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ and Property 1(d) gives

$$\left|\chi_l(t+\kappa)-\chi_l(t)\right|^2 \le 3\left|\left[\Phi_l(\kappa)-E_3\right]\Phi_l(t)\chi_{l0}\right|^2+3\left|\int_0^t\left[\Phi_l(\kappa)-E_3\right]\Phi_l(t-s)\,\varpi_l(s)\,ds\right|^2+3\left|\int_t^{t+\kappa}\Phi_l(t+\kappa-s)\,\varpi_l(s)\,ds\right|^2$$

$$\le 3\left\|\Phi_l(\kappa)-E_3\right\|^2 e^{2\beta t}\left|\chi_{l0}\right|^2+3\left\|\Phi_l(\kappa)-E_3\right\|^2\left(\int_0^t e^{2\beta(t-s)}\left|\varpi_l(s)\right|ds\right)^2+3\left(\int_t^{t+\kappa}e^{\beta(t+\kappa-s)}\left|\varpi_l(s)\right|ds\right)^2$$

$$=3\left\|\Phi_l(\kappa)-E_3\right\|^2\left[e^{2\beta t}\left|\chi_{l0}\right|^2+\left(\int_0^t e^{2\beta(t-s)}\left|\varpi_l(s)\right|ds\right)^2\right]+3\left(\int_t^{t+\kappa}e^{\beta(t+\kappa-s)}\left|\varpi_l(s)\right|ds\right)^2$$

$$\le 3\left\|\Phi_l(\kappa)-E_3\right\|^2\left[e^{2\beta t}\left|\chi_{l0}\right|^2+\varphi(t)\int_0^t\left|\varpi_l(s)\right|^2 ds\right]+3\left(\int_t^{t+\kappa}e^{\beta(t+\kappa-s)}\left|\varpi_l(s)\right|ds\right)^2$$

where we have used the Cauchy-Schwarz inequality (9) in estimating the penultimate integral above. Hence,

$$\left\|\chi(t+\kappa)-\chi(t)\right\|^2=\sum_{l=1}^{\infty}\left|\chi_l(t+\kappa)-\chi_l(t)\right|^2\le J+K \tag{14}$$

where

$$J=3\sum_{l=1}^{\infty}\left\|\Phi_l(\kappa)-E_3\right\|^2\left[e^{2\beta t}\left|\chi_{l0}\right|^2+\varphi(t)\int_0^t\left|\varpi_l(s)\right|^2 ds\right], \tag{15}$$

$$K=3\sum_{l=1}^{\infty}\left(\int_t^{t+\kappa}e^{\beta(t+\kappa-s)}\left|\varpi_l(s)\right|ds\right)^2 \tag{16}$$

To estimate J in (15), we split J into two: the tail series J_1 and the partial sum J_2, given as

$$J_1=3\sum_{l=N+1}^{\infty}\left\|\Phi_l(\kappa)-E_3\right\|^2\left[e^{2\beta t}\left|\chi_{l0}\right|^2+\varphi(t)\int_0^t\left|\varpi_l(s)\right|^2 ds\right] \tag{17}$$

$$J_2=3\sum_{l=1}^{N}\left\|\Phi_l(\kappa)-E_3\right\|^2\left[e^{2\beta t}\left|\chi_{l0}\right|^2+\varphi(t)\int_0^t\left|\varpi_l(s)\right|^2 ds\right] \tag{18}$$

respectively, and $N>0$ is an integer to be suitably chosen.
We first estimate J_1. Applying Property 1(d) on equation (17), we obtain

$$J_1=3\sum_{l=N+1}^{\infty}\left\|\Phi_l(\kappa)-E_3\right\|^2\left[e^{2\beta t}\left|\chi_{l0}\right|^2+\varphi(t)\int_0^t\left|\varpi_l(s)\right|^2 ds\right]$$
$$=3\psi^2(\kappa)\sum_{l=N+1}^{\infty}\left[e^{2\beta t}\left|\chi_{l0}\right|^2+\varphi(t)\int_0^t\left|\varpi_l(s)\right|^2 ds\right] \tag{19}$$

Noting that $t\in[0,\ T]$, we can reduce (19) to

$$J_1=3\psi^2(\kappa)\sum_{l=N+1}^{\infty}\left[e^{2\beta T}\left|\chi_{l0}\right|^2+\varphi(T)\int_0^T\left|\varpi_l(s)\right|^2 ds\right] \tag{20}$$

The convergence of the series in equation (20) can be readily deduced from the convergence of both $\sum_{l=1}^{\infty}\left|\chi_{l0}\right|^2$and $\sum_{l=1}^{\infty}\int_0^T\left|\varpi_l(s)\right|^2 ds$. Consequently, for any given $\varepsilon>0$, we can select $N>0$ in order that $J_1<\frac{1}{3}\varepsilon$.
Next, the estimation of J_2 follows below:

$$J_2=3\sum_{l=1}^{N}\left\|\Phi_l(\kappa)-E_3\right\|^2\left[e^{2\beta t}\left|\chi_{l0}\right|^2+\varphi(t)\int_0^t\left|\varpi_l(s)\right|^2\mathrm{d}s\right] \tag{21}$$

Observe from definition that as $\kappa\to 0$ for all l, $\left\|\Phi_l(\kappa)-E_3\right\|^2\to 0$. Again, since the sum on the RHS of equation (21) contains finitely many summands; so, whenever $|\kappa|<\delta_1$ can make a choice of δ_1 in order that $J_2<\frac{1}{3}\varepsilon$.

Finally, we estimate K. Using the Cauchy-Schwarz inequality

$$\left(\int_t^{t+\kappa}e^{\beta(t+\kappa-s)}\cdot\left|\varpi_l(s)\right|ds\right)^2\le\int_t^{t+\kappa}e^{2\beta(t+\kappa-s)}ds\cdot\int_t^{t+\kappa}\left|\varpi_l(s)\right|^2ds$$
$$=\varphi(\kappa)\int_t^{t+\kappa}\left|\varpi_l(s)\right|^2ds \tag{22}$$

on the RHS of (16), we find that

$$K\le 3\sum_{l=1}^{\infty}\varphi(\kappa)\int_t^{t+\kappa}\left|\varpi_l(s)\right|^2ds\le 3\varphi(\kappa)\sum_{l=1}^{\infty}\int_0^T\left|\varpi_l(s)\right|^2ds$$
$$\le 3\varphi(\kappa)\rho_0^2$$

Clearly, we can select δ_2 in such a way that $K<\frac{1}{3}\varepsilon$ whenever $|\kappa|<\delta_2$. Hence, choosing $\delta=\min\{\delta_1,\delta_2\}$, we see that it is possible to make the expression $\|\chi(t+\kappa)-\chi(t)\|^2$ in (14) smaller than any given $\varepsilon>0$.

Next, we need to examine the second case $\|\chi(t+\kappa)-\chi(t)\|$ for $\kappa<0$. Alternatively, for any given $\kappa>0$, let us consider the expression $\|\chi(t)-\chi(t-\kappa)\|$. Using (5), we get that

$$\chi_l(t)-\chi_l(t-\kappa)=\Phi_l(t)\chi_{l0}+\int_0^t\Phi_l(t-s)\varpi_l(s)ds-\left[\Phi_l(t-\kappa)\chi_{l0}+\int_0^{t-\kappa}\Phi_l(t-\kappa-s)\varpi_l(s)ds\right]$$
$$=\Phi_l(t)\chi_{l0}-\Phi_l(t-\kappa)\chi_{l0}+\int_0^t\Phi_l(t-s)\varpi_l(s)ds-\int_0^{t-\kappa}\Phi_l(t-\kappa-s)\varpi_l(s)ds$$
$$=(\Phi_l(\kappa)-E_3)\Phi_l(t-\kappa)\chi_{l0}+\int_0^{t-\kappa}\Phi_l(t-s)\varpi_l(s)ds+\int_{t-\kappa}^t\Phi_l(t-s)\varpi_l(s)ds-\int_0^{t-\kappa}\Phi_l(t-\kappa-s)\varpi_l(s)ds$$
$$=(\Phi_l(\kappa)-E_3)\Phi_l(t-\kappa)\chi_{l0}+\int_0^{t-\kappa}(\Phi_l(\kappa)-E_3)\Phi_l(t-\kappa-s)\varpi_l(s)ds+\int_{t-\kappa}^t\Phi_l(t-s)\varpi_l(s)ds$$

Using a method analogous to the way we estimated $\|\chi(t+\kappa)-\chi(t)\|^2$, we can demonstrate that for all given $\varepsilon>0$, a corresponding $\delta>0$ exists such that $\|\chi(t)-\chi(t-\kappa)\|<\varepsilon$ whenever $|\kappa|<\delta$. Consequently, we conclude that $\chi(t)$ is continuous for all $t\in[0,T]$. With this result, the lemma is now fully proven.

Combining Lemmas 1 and 2, we have thus proved the main result in this article, contained in the following theorem.

Theorem 1: For any given $T > 0$, a unique solution to the infinite system of three-coupled ODE given in (4) exists in $C(0, T; \ell_2)$ where $\varpi(\cdot) \in S(\rho_0)$, $0 \leq \mu_l \leq \mu$ and $0 \leq \lambda_l \leq \lambda$, $l = 1, 2, \ldots$.

CONCLUSION

In this research, we established that a unique solution to the infinite system of three-coupled ODE (4) exists in $C(0, T; \ell_2)$. The findings of this research hold a lot of promising implications to the study of control and differential games modelled by infinite system of coupled ODEs. This can be seen from below.

After Ibragimov et al. (2008) showed that a unique solution for the infinite two-coupled system (1) exists in the ℓ_2-space for the case $\alpha_l \geq 0$, Ibragimov and Kuchkarova (2021) extended their result to cover the case $\alpha_l < 0$. This existence and uniqueness result of Ibragimov and Kuchkarova (2021) allowed Ibragimov et al. (2022b) to study the control and differential game modeled by (1) for the case $\alpha_l \leq 0$ with the control function of the pursuer and evader subjected to integral constraints; Ibragimov et al. (2022b) then obtained sufficient conditions to drive the state of the system from initial state to the origin of ℓ_2-space, optimal strategies for the players and the optimal pursuit time in the game.

Madhavan et al. (2024a) proved that a unique solution to the three-coupled ODE in an infinite system (2) exists in the $C(0, T; \ell_2)$ -space, for only the case $\alpha_l, \beta_l \geq 0$. The findings in this paper have effectively extended their result to the case $\alpha_l, \beta_l < 0$ in the system (2).

Also, the existence and uniqueness result of Madhavan et al. (2024a) allowed Madhavan et al. (2024b) to investigate a control and pursuit game problem modelled by three-coupled ODEs in an infinite system (2) with integral constraints imposed on the control function of the players. As an implication of our finding, this research's existence and uniqueness result permit that differential game and control problems be investigated for games modelled by infinite systems of three-coupled ODE (4).

For further research, the existence of a unique solution of three-coupled ODEs in an infinite system (4) for different values of the parameters μ_l, λ_l not yet considered in the literature, can be studied. Generalization of our result to show the existence of a unique solution in $C(0, T; \ell_2)$ of n-coupled ODE in an infinite system with suitable conditions on the coefficient matrix of the system can also be undertaken.

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