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Diagonally Multistep Block Method for Solving Volterra Integro-Differential Equation with Delay

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ABSTRACT

Volterra integro-differential equation with delay (VIDED) is solved using a diagonally multistep block method (DMB). This study provides the derivation of the DMB utilizing Taylor series with a constant step size strategy for treating the first order VIDED. In predictor-corrector mode, the DMB method combines the predictor and corrector formulae. It approximates two numerical solutions simultaneously within a block. The algorithm for the approximation solution is developed and the Newton-Cotes formulae are adapted in the DMB method to estimate the solution for an integral component. Theoretically, the consistency and zero stability that led to convergence properties are examined. The stability region also has been plotted. The numerical results indicate that the developed method is superior in terms of the number of steps, accuracy and computation time taken.

Keywords: Volterra integro-differential equation with delay, diagonally multistep block, Newton-Cotes formulae

INTRODUCTION

The following Volterra integro-differential equation with delay is considered:

$$y'(t) = F\left(t, y(t), y(t-\tau)), \int_{t-\tau}^{t} K(t,u,y(u),y(u-\tau))du\right), \qquad t \in [t_0, T]. \tag{1}$$

where

$$z(t) = \int_0^t K(t,u,y(u),y(u-\tau))\ du,$$

depending on the initial function,

$$y(t) = \phi(t), \text{ where } t \in [t_0 - \tau, t_0]. \tag{2}$$

F, K and ϕ are supposed to be sufficiently smooth. In this case, the delay term, τ, is assumed to be continuous and positive integer. Equation (1) can be simplified to a standard initial value problem to yield an approximation of a numerical solution as follows,

$$y'(t) = F(t, y(t), y(t-\tau), z(t)) \tag{3}$$

where

$$z(t) = \int_0^t K(t,u,y(u),y(u-\tau))du. \tag{4}$$

On mesh points with $t_0 = a,\ t_k = T,\ t_0 < t_1 < \ldots < t_k$, the block methods have been developed to approximate the solutions of the standard initial value problem. Frequently, the constant step size, h, is utilized to analyse these methods.

This type of equation arises widely in steady-state solution (Verdugo, 2018) and HIV-1 infection model (Ali et al., 2018) which the delay term denotes the dormant period. The analytical solution for the VIDED is too complex; thus, the qualitative results may depend on numerical approaches. Brunner and Zhang (1999) obtained several primary discontinuity results for integral and integro-differential equations (IDE) involving different delays. Meanwhile, Ayad (2001) came up with the idea of using a polynomial spline function to obtain an approximation of the solution to two distinct kinds of IDE with delay, namely Fredholm IDE with delay and VIDED.

Schoenberg was the first to introduce the B-spline in 1949. B-splines are the typical non-linear smooth geometry representation in numerical computation. Ali (2009) established a numerical solution to the VIDED problem using the expansion method (collocation and partition) in conjunction with basis function of B-spline polynomials. Meanwhile, Salih et al. (2010), had solved the n^{th} -order linear VIDED with convolution type using the Galerkin method with the B-spline function. Salih et al. (2014) created a MATLAB algorithm to evaluate the n^{th} order linear VIDED of convolution types and employed the B-spline function with the aid of the Weddle rule to estimate the required integrals for the equations. Galerkin's method is prevalent in solving differential equations numerically. Consequently, Zaidan (2012) introduced Bernstein polynomial of degree two defined as the weighted residual method with Galerkin's method in solving the linear VIDED.

Yüzbaşı and Karaçayır (2018) approximated a solution for higher order VIDED by converting the problem into a linear algebraic equation system. They suggested a Galerkin-like approach to approximate this system. Baharum et al. (2022) solved VIDED using the third order multistep block method. While Janodi et al. (2020) considered a hybrid block method when solving Volterra IDE. Lagrange interpolation polynomials have been used to solve the delay in Ismail et al. (2020). Moreover, Baharum et al. (2022) implement the Boole's rule strategy in solving the Volterra IDE using the multistep block method.

The two-point diagonally multistep block approach is used in this study to obtain new numerical findings for Volterra integro-differential equation with delay problem. Several examples illustrate that the proposed method yields relevant numerical results.

DERIVATION OF THE METHOD

Linear difference operator, L associated with

$$L[y(t);h]=\sum_{i=0}^{k}\alpha_i y(t+ih)-h\sum_{i=0}^{k}\beta_i y'(t+ih), \tag{5}$$

as $y(t)$ is a function and continuously differentiable on $[t_0,T]$. Linear difference operator is generated by substituting $y(t)$ and its derivatives, $y'(t)$ with Taylor series.

$$L[y(t);h]=\sum_{i=0}^{k}\alpha_i\left(y(t)+ihy'(t)+\frac{i^2}{2!}h^2y''(t)\right)-h\sum_{i=0}^{k}\beta_i\left(y'(t)+ihy''(t)+\frac{i^2}{2!}h^2y'''(t)\right). \tag{6}$$

DMB method has been formed by evaluating $y(t+h)$ and $y(t+2h)$ with delay arguments and solutions, respectively. As indicated below, the first-point corrector formula was developed by expanding the linear multistep method, (LMM) (5).

$$\sum_{i=3}^{k}\alpha_i y(t+ih)=h\sum_{i=0}^{k-4}\beta_i y'(t+ih)+h\sum_{i=2}^{k}\beta_i y'(t+ih).$$

When the number of steps, $k=4$, DMB method formulated the first point corrector formula;

$$\sum_{i=3}^{4}\alpha_i y(t+ih)=h\sum_{i=0}^{0}\beta_i y'(t+ih)+h\sum_{i=2}^{4}\beta_i y'(t+ih),$$

$$\alpha_3 y(t+3h)+\alpha_4 y(t+4h)=h\beta_0 y'(t)+h\beta_2 y'(t+2h)+h\beta_3 y'(t+3h)+h\beta_4 y'(t+4h).$$

By letting $\alpha_3 = -1$ and $\alpha_4 = 1$ yields,

$$-y(t+3h) + y(t+4h) = h\beta_0 y'(t) + h\beta_2 y'(t+2h) + h\beta_3 y'(t+3h) + h\beta_4 y'(t+4h),$$

$$y(t+4h) = y(t+3h) + h\beta_0 y'(t) + h\beta_2 y'(t+2h) + h\beta_3 y'(t+3h) + h\beta_4 y'(t+4h). \qquad (7)$$

Taylor series produces the expression for $y(t+4h)$, $y(t+3h)$, $y'(t)$, $y'(t+2h)$, $y'(t+3h)$, and $y'(t+4h)$.

$$y(t+4h) = y(t) + 4hy'(t) + \frac{16}{2}h^2 y''(t) + \frac{64}{6}h^3 y'''(t) + \frac{256}{24}h^4 y^{(4)}(t),$$

$$y(t+3h) = y(t) + 3hy'(t) + \frac{9}{2}h^2 y''(t) + \frac{27}{6}h^3 y'''(t) + \frac{81}{24}h^4 y^{(4)}(t),$$

$$y'(t+2h) = y'(t) + 2hy''(t) + \frac{4}{2}h^2 y'''(t) + \frac{8}{6}h^3 y^{(4)}(t),$$

$$y'(t+3h) = y'(t) + 3hy''(t) + \frac{9}{2}h^2 y'''(t) + \frac{27}{6}h^3 y^{(4)}(t),$$

$$y'(t+4h) = y'(t) + 4hy''(t) + \frac{16}{2}h^2 y'''(t) + \frac{64}{6}h^3 y^{(4)}(t).$$

The terms of Taylor series are truncated as the fourth-order method at the fourth derivative. Hence, Taylor series is substituted into the equation (7) and yields,

$$\begin{aligned} & y(t) + 4hy'(t) + \frac{16}{2}h^2 y''(t) + \frac{64}{6}h^3 y'''(t) + \frac{256}{24}h^4 y^{(4)}(t) \\ & \cong y(t) + hy'(t)\left(3 + \beta_0 + \beta_2 + \beta_3 + \beta_4\right) + h^2 y''(t)\left(\frac{9}{2} + 2\beta_2 + 3\beta_3 + 4\beta_4\right) \\ & \quad + h^3 y'''(t)\left(\frac{27}{6} + 2\beta_2 + \frac{9}{2}\beta_3 + \frac{16}{2}\beta_4\right) + h^4 y^{(4)}(t)\left(\frac{81}{24} + \frac{8}{6}\beta_2 + \frac{27}{6}\beta_3 + \frac{64}{26}\beta_4\right). \end{aligned} \qquad (8)$$

As a result of associating the left and right sides of equation (8), the result would be,

$$3 + \beta_0 + \beta_2 + \beta_3 + \beta_4 = 4,$$

$$\frac{9}{2} + 2\beta_2 + 3\beta_3 + 4\beta_4 = \frac{16}{2},$$

$$\frac{27}{6} + 2\beta_2 + \frac{9}{2}\beta_3 + \frac{16}{2}\beta_4 = \frac{64}{6},$$

$$\frac{81}{24} + \frac{8}{6}\beta_2 + \frac{27}{6}\beta_3 + \frac{64}{26}\beta_4 = \frac{256}{24}.$$

The coefficients β_i obtained as follows,

$$\beta_0 = \frac{1}{96}, \quad \beta_2 = -\frac{7}{48}, \quad \beta_3 = \frac{3}{4}, \quad \beta_4 = \frac{37}{96}.$$

Hence, the formula could be expressed as

$$y_{n+4} = y_{n+3} + h\left(\frac{37}{96}F_{n+4} + \frac{3}{4}F_{n+3} - \frac{7}{48}F_{n+2} + \frac{1}{96}F_n\right).$$

By choosing the value $n = n-3$, the first point of corrector formula for DBM method is obtained,

$$y_{n+1} = y_n + h\left(\frac{37}{96}F_{n+1} + \frac{3}{4}F_n - \frac{7}{48}F_{n-1} + \frac{1}{96}F_{n-3}\right).$$

Calculated based on the LMM, the corrector formula for the second point of the DBM method yields,

$$\sum_{i=3}^{k-2}\alpha_i y(t+ih)+\sum_{i=k}^{k}\alpha_i y(t+ih)=h\sum_{i=0}^{k-5}\beta_i y'(t+ih)+h\sum_{i=2}^{k-3}\beta_i y'(t+ih)+h\sum_{i=4}^{k}\beta_i y'(t+ih).$$

With $k=5$ as the step number, the following will be:

$$\sum_{i=3}^{3}\alpha_i y(t+ih)+\sum_{i=5}^{5}\alpha_i y(t+ih)=h\sum_{i=0}^{0}\beta_i y'(t+ih)+h\sum_{i=2}^{2}\beta_i y'(t+ih)+h\sum_{i=4}^{5}\beta_i y'(t+ih),$$

$$\alpha_3 y(t+3h)+\alpha_5 y(t+5h)=h\beta_0 y'(t)+h\beta_2 y'(t+2h)+h\beta_4 y'(t+4h)+h\beta_5 y'(t+5h).$$

Hence, letting $\alpha_3=-1$ and $\alpha_5=1$,

$$-y(t+3h)+y(t+5h)=h\beta_0 y'(t)+h\beta_2 y'(t+2h)+h\beta_4 y'(t+4h)+h\beta_5 y'(t+5h),$$

$$y(t+5h)=y(t+3h)+h\beta_0 y'(t)+h\beta_2 y'(t+2h)+h\beta_4 y'(t+4h)+h\beta_5 y'(t+5h). \quad (9)$$

Taylor series will expand $y(x)$ and $y'(x)$.

$$y(t+5h)=y(t)+5hy'(t)+\frac{25}{2}h^2y''(t)+\frac{125}{6}h^3y'''(t)+\frac{625}{24}h^4y^{(4)}(t),$$

$$y(t+3h)=y(t)+3hy'(t)+\frac{9}{2}h^2y''(t)+\frac{27}{6}h^3y'''(t)+\frac{81}{24}h^4y^{(4)}(t),$$

$$y'(t+2h)=y'(t)+2hy''(t)+\frac{4}{2}h^2y'''(t)+\frac{8}{6}h^3y^{(4)}(t),$$

$$y'(t+4h)=y'(t)+4hy''(t)+\frac{16}{2}h^2y'''(t)+\frac{64}{6}h^3y^{(4)}(t),$$

$$y'(t+5h)=y'(t)+5hy''(t)+\frac{25}{2}h^2y'''(t)+\frac{125}{6}h^3y^{(4)}(t).$$

Since the method order is four, the terms of Taylor series are truncated at the fourth derivative. Hence, replacing the expansion for the equation (9) and acquiring,

$$\begin{aligned}&y(t)+5hy'(t)+\frac{25}{2}h^2y''(t)+\frac{125}{6}h^3y'''(t)+\frac{625}{24}h^4y^{(4)}(t)\\ &\cong y(t)+hy'(t)\left(3+\beta_0+\beta_2+\beta_4+\beta_5\right)+h^2y''(t)\left(\frac{9}{2}+2\beta_2+4\beta_4+5\beta_5\right)\\ &+h^3y'''(t)\left(\frac{27}{6}+2\beta_2+\frac{16}{2}\beta_4+\frac{25}{2}\beta_5\right)+h^4y^{(4)}(t)\left(\frac{81}{24}+\frac{8}{6}\beta_2+\frac{64}{6}\beta_4+\frac{125}{6}\beta_5\right).\end{aligned} \quad (10)$$

Associating the left and right sides of the equation (10) could yield,

$$3+\beta_0+\beta_2+\beta_4+\beta_5=5,$$

$$\frac{9}{2}+2\beta_2+4\beta_4+5\beta_5=\frac{25}{2},$$

$$\frac{27}{6}+2\beta_2+\frac{16}{2}\beta_4+\frac{25}{2}\beta_5=\frac{125}{6},$$

$$\frac{81}{24}+\frac{8}{6}\beta_2+\frac{64}{6}\beta_4+\frac{125}{6}\beta_5=\frac{625}{24}.$$

Thus,

$$\beta_0=-\frac{1}{60},\quad \beta_2=\frac{1}{6},\quad \beta_4=\frac{19}{12},\quad \beta_5=\frac{4}{15}.$$

The formula can be written as

$$y_{n+5} = y_{n+3} + h\left(\frac{4}{15}F_{n+5} + \frac{19}{12}F_{n+4} + \frac{1}{6}F_{n+2} - \frac{1}{60}F_n\right).$$

After letting $n = n-3$, the corrector formula for the second point of the DBM method can be obtained as follows,

$$y_{n+2} = y_n + h\left(\frac{4}{15}F_{n+2} + \frac{19}{12}F_{n+1} + \frac{1}{6}F_{n-1} - \frac{1}{60}F_{n-3}\right).$$

The corrector formula of DMB:

$$y_{n+1} = y_n + h\left(\frac{1}{96}F_{n-3} - \frac{7}{48}F_{n-1} + \frac{3}{4}F_n + \frac{37}{96}F_{n+1}\right),$$

$$y_{n+2} = y_n + h\left(-\frac{1}{60}F_{n-3} + \frac{1}{6}F_{n-1} + \frac{19}{12}F_{n+1} + \frac{4}{15}F_{n+2}\right).$$

(11)

The first point predictor formula based on the LMM, can be created using the same procedure as follows,

$$\sum_{i=3}^{4}\alpha_i y(t+ih) = h\sum_{i=0}^{2}\beta_i y'(t+ih),$$

and for the predictor formula's second point,

$$\sum_{i=3}^{3}\alpha_i y(t+ih) + \sum_{i=5}^{5}\alpha_i y(t+ih) = h\sum_{i=0}^{2}\beta_i y'(t+ih).$$

Hence, the formulae for predictor would be identified as follows:

$$\begin{aligned} y_{n+1} &= y_n + h\left(\frac{53}{12}F_{n-1} - \frac{16}{3}F_{n-2} + \frac{23}{12}F_{n-3}\right), \\ y_{n+2} &= y_n + h\left(\frac{37}{3}F_{n-1} - \frac{50}{3}F_{n-2} + \frac{19}{3}F_{n-3}\right). \end{aligned} \qquad (12)$$

ANALYSIS OF THE METHOD

Using the matrix difference equation defined below, the order of the proposed method could be determined,

$$\alpha Y_N = \beta h F_N, \qquad (13)$$

where,

$$Y_N = \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \\ y_{n+1} \\ y_{n+2} \end{bmatrix}, \qquad F_N = \begin{bmatrix} F_{n-3} \\ F_{n-2} \\ F_{n-1} \\ F_n \\ F_{n+1} \\ F_{n+2} \end{bmatrix}.$$

The DMB method indicated in the form of a linear multistep method (5). The equation (6) can be represented in its general form as

$$L[y(t);h] = C_0 y(t) + C_1 h y'(t) + C_2 h^2 y''(t) + \ldots + C_p h^p y^{(p)}(t) + \ldots,$$

whereas,

$$C_p = \sum_{d=0}^{k} \frac{d^p \alpha_d}{p!} - \sum_{d=0}^{k} \frac{d^{(p-1)} \beta_d}{(p-1)!}, \quad p = 0,1,2,\ldots, \tag{14}$$

The vector columns of the matrices, α_d and β_d generated by the DMB method.

Definition 1.
When $C_0 = C_1 = \ldots = C_{p-1} = C_p = 0$ and $C_{p+1} \neq 0$, the LMM takes p as the order of the method.

Thus, the developed method is written in matrix difference form as (13),

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \\ y_{n+1} \\ y_{n+2} \end{bmatrix} = h \begin{bmatrix} \frac{1}{96} & 0 & -\frac{7}{48} & \frac{3}{4} & \frac{37}{96} & 0 \\ -\frac{1}{60} & 0 & \frac{1}{6} & 0 & \frac{19}{12} & \frac{4}{15} \end{bmatrix} \begin{bmatrix} F_{n-3} \\ F_{n-2} \\ F_{n-1} \\ F_n \\ F_{n+1} \\ F_{n+2} \end{bmatrix}.$$

Equation (14) is used to identify the order of method,

$$C_0 = \sum_{d=0}^{5} \frac{d^0}{0!} \alpha_d = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$C_1 = \sum_{d=0}^{5} \frac{d^1 \alpha_d}{1!} - \sum_{d=0}^{5} \frac{d^0 \beta_d}{0!},$$
$$= \frac{1}{1!}\left(1\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$
$$-\frac{1}{0!}\left(\begin{bmatrix} \frac{1}{96} \\ -\frac{1}{60} \end{bmatrix} + \begin{bmatrix} 0 \\ [0 \end{bmatrix} + \begin{bmatrix} -\frac{7}{48} \\ \frac{1}{6} \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{37}{96} \\ \frac{19}{12} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{4}{15} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$C_2 = \sum_{d=0}^{5} \frac{d^2 \alpha_d}{2!} - \sum_{d=0}^{5} \frac{d^1 \beta_d}{1!},$$
$$= \frac{1}{2!}\left(1^2\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2^2\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 3^2\begin{bmatrix} -1 \\ -1 \end{bmatrix} + 4^2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5^2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$
$$-\frac{1}{1!}\left(1\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -\frac{7}{48} \\ \frac{1}{6} \end{bmatrix} + 3\begin{bmatrix} \frac{3}{4} \\ 0 \end{bmatrix} + 4\begin{bmatrix} \frac{37}{96} \\ \frac{19}{12} \end{bmatrix} + 5\begin{bmatrix} 0 \\ \frac{4}{15} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$C_3=\sum_{d=0}^{5}\frac{d^3\alpha_d}{3!}-\sum_{d=0}^{5}\frac{d^2\beta_d}{2!},$$
$$=\frac{1}{3!}\left(1^3\begin{bmatrix}0\\0\end{bmatrix}+2^3\begin{bmatrix}0\\0\end{bmatrix}+3^3\begin{bmatrix}-1\\-1\end{bmatrix}+4^3\begin{bmatrix}1\\0\end{bmatrix}+5^3\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$-\frac{1}{2!}\left(1^2\begin{bmatrix}0\\0\end{bmatrix}+2^2\begin{bmatrix}-\frac{7}{48}\\\frac{1}{6}\end{bmatrix}+3^2\begin{bmatrix}\frac{3}{4}\\0\end{bmatrix}+4^2\begin{bmatrix}\frac{37}{96}\\\frac{19}{12}\end{bmatrix}+5^2\begin{bmatrix}0\\\frac{4}{15}\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}.$$

$$C_4=\sum_{d=0}^{5}\frac{d^4\alpha_d}{4!}-\sum_{d=0}^{5}\frac{d^3\beta_d}{3!},$$
$$=\frac{1}{4!}\left(1^4\begin{bmatrix}0\\0\end{bmatrix}+2^4\begin{bmatrix}0\\0\end{bmatrix}+3^4\begin{bmatrix}-1\\-1\end{bmatrix}+4^4\begin{bmatrix}1\\0\end{bmatrix}+5^4\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$-\frac{1}{3!}\left(1^3\begin{bmatrix}0\\0\end{bmatrix}+2^3\begin{bmatrix}-\frac{7}{48}\\\frac{1}{6}\end{bmatrix}+3^3\begin{bmatrix}\frac{3}{4}\\0\end{bmatrix}+4^3\begin{bmatrix}\frac{37}{96}\\\frac{19}{12}\end{bmatrix}+5^3\begin{bmatrix}0\\\frac{4}{15}\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}.$$

$$C_5=\sum_{d=0}^{5}\frac{d^5\alpha_d}{5!}-\sum_{d=0}^{5}\frac{d^4\beta_d}{4!},$$
$$=\frac{1}{5!}\left(1^5\begin{bmatrix}0\\0\end{bmatrix}+2^5\begin{bmatrix}0\\0\end{bmatrix}+3^5\begin{bmatrix}-1\\-1\end{bmatrix}+4^5\begin{bmatrix}1\\0\end{bmatrix}+5^5\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$-\frac{1}{4!}\left(1^4\begin{bmatrix}0\\0\end{bmatrix}+2^4\begin{bmatrix}-\frac{7}{48}\\\frac{1}{6}\end{bmatrix}+3^4\begin{bmatrix}\frac{3}{4}\\0\end{bmatrix}+4^4\begin{bmatrix}\frac{37}{96}\\\frac{19}{12}\end{bmatrix}+5^4\begin{bmatrix}0\\\frac{4}{15}\end{bmatrix}\right)=\begin{bmatrix}-\frac{53}{1440}\\\frac{13}{180}\end{bmatrix}.$$

while $C_5\neq 0.$

$$\therefore C_{p+1}=C_5=\begin{bmatrix}-\frac{53}{1440} & \frac{13}{180}\end{bmatrix}^T.$$

By referring to Definition 1, the DMB method has order four, with C_5 being the error constant vector. In the similar procedure, by checking the order for the predictor formula in (12), the calculation process revealed that $C_0=C_1=C_2=C_3=0$ and $C_4=C_{p+1}\neq 0.$ This conclude that the predictor formula satisfies the order three condition and the error constant is $C_4=\begin{bmatrix}-\frac{55}{24} & 9\end{bmatrix}^T$.

Definition 2.
The method is considered as consistent if has at least one order method.

Since the DBM method is fourth order method, thus this method can be concluded as consistent.

Definition 3.
If the first characteristics polynomial $\rho(r)$ have roots such as $|r_j| \leq 1$ and the multiplicity of the roots must not exceed two, the method can be considered as zero stable,

$$\rho(r) = \det |\sum_{j=0}^{k} A^{(j)} r^{(k-j)}| = 0$$

The DMB method could be expressed in matrix form as follows:

$$A^0 Y_M - A^1 Y_{M-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_N \end{bmatrix} = 0.$$

The following are the first characteristics polynomials to describe the zero stability:

$$\rho(r) = \det |A^0 r - A^1| = \det \left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} r - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right| = r(r-1).$$

The roots *r* is $|r_j| \leq 1,$ hence the method is zero stable.

Definition 4.
It is said that a method is convergent if it is consistent and zero-stable.

DMB has been proven to be converging as well as zero-stable and consistent.

IMPLEMENTATION

This study uses three components for solving the VIDED, i.e., initial value problem, delay solution and integral part of the VIDED. The DMB, which is built on predictor-corrector formulae in *PE(CE)* mode, will be used to approximate the two-point solutions simultaneously. Before complying with the proposed method, it is necessary to approximate the initial points.

This study considers the problems of VIDED with constant delay type. The proposed method is implemented in C code to approximate the numerical solutions. When determining the delay solution, it may require locating the delay arguments. The delay solution, $y(t-\tau)$ depends on where $(t-\tau)$ is located. Since the implementation is constant step size, the location could be associated with the previously approximated solution. In addition, the initial function, $\phi(t)$ computes $y(t-\tau)$ if $(t-\tau) \leq t_0$. The approximate solution for VIDED is computed using the DMB method for solving the ODE component of the VIDED.

Since explicitly solving VIDED is impractical, the specific Newton-Cotes rules have been associated with dealing with the integral component of this equation. The formula used by the Newton-Cotes rule is the composite Simpson rule, which could be expressed in the form of:

$$z_{n+1} = \frac{h}{3} \sum_{i=0}^{n+1} \omega_i^s K(t_{n+1}, t_i, y_i),$$

$$z_{n+2} = \frac{h}{3} \sum_{i=0}^{n+1} \omega_i^s K(t_{n+1}, t_i, y_i) + \frac{h}{6} \left(K(t_{n+2}, t_{n+1}, y_{n+1}) + 4K(t_{n+2}, t_{n+\frac{3}{2}}, y_{n+\frac{3}{2}}) + K(t_{n+2}, t_{n+2}, y_{n+2}) \right),$$

where ω_i^s are Simpson's rule weight, $1, 4, 2, 4, \ldots, 2, 4, 1$. The undetermined evaluate of $y_{n+\frac{3}{2}}$ is estimated via quadratic interpolation.

$$y_{n+\frac{3}{2}} = \frac{1}{16} y_{n-1} - \frac{5}{16} y_n + \frac{15}{16} y_{n+1} + \frac{5}{16} y_{n+2}.$$

As mentioned in the algorithm below, the procedure is repeated until the end of interval.

ALGORITHM

Step 1 : Set
$N,\ t_0 = a,\ t_n = b,\ h = \frac{b-a}{N},\ y_0 = y(t_0),\ z_0 = z(t_0, y(t_0)).$

Step 2 : When $n = 0$,
Calculate $y(t-\tau)$ and F_0.

Step 3 : When $n = 1,\ 2,\ 3$,
Calculate $t_n = t_0 + nh$, and evaluate the initial values with Runge-Kutta method and Simpson's $1/3$ rule.
Calculate $y(t_n - \tau)$ and F_n.

Step 4 : When $n = 3,\ 5,\ 7,\ ...$
For $i = 1,\ 2$,
Evaluate $t_{n+i} = t_n + ih$,

Step 5 : Determine the approximate predictor value for y_{n+i}^{p} using the derived predictor formulae of DMB method in $PE(CE)$ mode.
: Compute $y(t-\tau)$.
: To approximate the integral part, apply the composite Simpson's rule.
: Calculate F_{n+i}^{p}.

Step 6 : Calculate the approximate solution, y_{n+i}^{c} using the derived corrector formulae of DMB method in $PE(CE)$ mode.
: Calculate $y(t-\tau)$.
: To solve the integral, use the composite Simpson's rule.
: Calculate F_{n+i}^{c}.

Step 7 : Go to **Step 4** and repeat until N.

Step 8 : OUTPUT: (t, y)

STABILITY REGION

The following test equation is used to obtain the stability region for the constant delay type:

$$y'(t) = \xi y(t-\tau) + \nu \int_0^t y(u)du. \qquad (15)$$

Assume $\tau = mh$, where the internal staged are not required to estimate $y(t-\tau)$ and substitute $y(t-\tau) = Y_{N-m}$. The expression of the DBM method in the matrix form given as,

$$\sum_{k=0}^{2} A_k Y_{N+k} = h \sum_{k=0}^{2} B_k F_{N+k},$$

$$A_0 Y_N + A_1 Y_{N+1} + A_2 Y_{N+2} = hB_0 F_N + hB_1 F_{N+1} + hB_2 F_{N+2}$$

where,

$$Y_N = \begin{bmatrix} y_{n-3} \\ y_{n-2} \end{bmatrix},\ Y_{N+1} = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix},\ Y_{N+2} = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix},\ F_N = \begin{bmatrix} F_{n-3} \\ F_{n-2} \end{bmatrix},\ F_{N+1} = \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix},\ F_{N+2} = \begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B_0 = \begin{bmatrix} \frac{1}{96} & 0 \\ -\frac{1}{60} & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} -\frac{7}{48} & \frac{3}{4} \\ \frac{1}{6} & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} \frac{37}{96} & 0 \\ \frac{19}{12} & \frac{4}{15} \end{bmatrix}.$$

From the test equation (15),

$$F_N = y'(t) = \xi y(t-\tau) + \nu \int_0^t y(u)du,$$

$$F_N = \xi Y_{N-m} + \nu \int_0^t y(u)du,$$

where the Simpson's quadrature rule will be insert at the integral part,

$$\int_0^t y(u)du = h\left(\frac{1}{3}Y_{N-2} + \frac{4}{3}Y_{N-1} + \frac{1}{3}Y_N\right).$$

Then, apply the test equation to the DMB method and obtain the result as follows,

$$\begin{aligned} A_0 Y_N + A_1 Y_{N+1} + A_2 Y_{N+2} = hB_0 &\left(\xi Y_{N-m} + \nu h\left(\frac{1}{3}Y_{N-2} + \frac{4}{3}Y_{N-1} + \frac{1}{3}Y_N\right)\right) \\ &+ hB_1\left(\xi Y_{N+1-m} + \nu h\left(\frac{1}{3}Y_{N-1} + \frac{4}{3}Y_N + \frac{1}{3}Y_{N+1}\right)\right) \\ &+ hB_2\left(\xi Y_{N+2-m} + \nu h\left(\frac{1}{3}Y_N + \frac{4}{3}Y_{N+1} + \frac{1}{3}Y_{N+2}\right)\right), \end{aligned}$$

Rearranging the equation yields

$$\begin{aligned} &\left(A_2 - \frac{1}{3}\nu h^2 B_2\right)Y_{N+2} + \left(A_1 - \frac{1}{3}\nu h^2 B_1 - \frac{4}{3}\nu h^2 B_2\right)Y_{N+1} + \left(A_0 - \frac{1}{3}\nu h^2 B_0 - \frac{4}{3}\nu h^2 B_1 - \frac{1}{3}\nu h^2 B_2\right)Y_N \\ &+\left(-\frac{4}{3}\nu h^2 B_0 - \frac{1}{3}\nu h^2 B_1\right)Y_{N-1} + \left(-\frac{1}{3}\nu h^2 B_0\right)Y_{N-2} - \xi h B_2 Y_{N+2-m} - \xi h B_1 Y_{N+1-m} - \xi h B_0 Y_{N-m} = 0. \end{aligned}$$

Replacing $H_1 = \xi h$ and $H_2 = \nu h^2$,

$$\begin{aligned} &\left(A_2 - \frac{1}{3}H_2 B_2\right)Y_{N+2} + \left(A_1 - \frac{1}{3}H_2 B_1 - \frac{4}{3}H_2 B_2\right)Y_{N+1} + \left(A_0 - \frac{1}{3}H_2 B_0 - \frac{4}{3}H_2 B_1 - \frac{1}{3}H_2 B_2\right)Y_N \\ &+\left(-\frac{4}{3}H_2 B_0 - \frac{1}{3}H_2 B_1\right)Y_{N-1} + \left(-\frac{1}{3}H_2 B_0\right)Y_{N-2} - H_1 B_2 Y_{N+2-m} - H_1 B_1 Y_{N+1-m} - H_1 B_0 Y_{N-m} = 0. \end{aligned}$$

The stability polynomial will determine as follows.

$$\begin{aligned} \pi(H_1, H_2; r) = \det \Bigg(&\left(A_2 - \frac{1}{3}H_2 B_2\right)r^{m+2} + \left(A_1 - \frac{1}{3}H_2 B_1 - \frac{4}{3}H_2 B_2\right)r^{m+1} \\ &+ \left(A_0 - \frac{1}{3}H_2 B_0 - \frac{4}{3}H_2 B_1 - \frac{1}{3}H_2 B_2\right)r^m + \left(-\frac{4}{3}H_2 B_0 - \frac{1}{3}H_2 B_1\right)r^{m-1} \\ &+ \left(-\frac{1}{3}H_2 B_0\right)r^{m-2} - H_1 B_2 r^2 - H_1 B_1 r^1 - H_1 B_0 r^0 \Bigg) = 0, \end{aligned}$$

$$\pi(H_1,H_2;r)=\frac{1}{720}r^{17}H_2^2+\frac{1}{80}rH_1^2-\frac{313}{1440}r^{24}H_2-\frac{1757}{1440}r^{23}H_2-\frac{311}{180}r^{22}H_2-\frac{313}{480}r^{14}H_1$$
$$+\frac{37}{3240}r^{24}H_2^2-\frac{97}{2160}r^{23}H_2^2-\frac{97}{108}r^{22}H_2^2-\frac{15997}{6480}r^{21}H_2^2-\frac{139}{108}r^{21}H_2$$
$$-\frac{1417}{1080}r^{20}H_2^2-\frac{103}{1440}r^{20}H_2-\frac{95}{432}r^{19}H_2^2+\frac{13}{1440}r^{19}H_2-\frac{1}{405}r^{18}H_2^2$$
$$-\frac{101}{96}r^{13}H_1+\frac{37}{360}r^{4}H_1^2-\frac{31}{96}r^{12}H_1-\frac{883}{720}r^{3}H_1^2+\frac{13}{480}r^{11}H_1-\frac{11}{90}r^{2}H_1^2$$
$$+\frac{37}{540}r^{14}H_1H_2-\frac{587}{1080}r^{13}H_1H_2-\frac{197}{60}r^{12}H_1H_2-\frac{613}{540}r^{11}H_1H_2-\frac{13}{270}r^{10}H_1H_2$$
$$+\frac{1}{120}r^{9}H_1H_2+r^{24}-r^{23}=0.$$

Figure 1 illustrates the stability region of the proposed method by replacing $r=\cos\theta+i\sin\theta$ where $0\le\theta\le 2\pi$ and $r=-1,\ 0,\ 1$ in the stability polynomial. Upon replacing $r=\cos\theta+i\sin\theta$, complex equation will be produced. The real and imaginary part have been solved simultaneously and determined the dot points that appeared in the region.

The shaded region represents the stable region for the proposed method while all the region outside the stable region is unstable. The stable region can be identified by identifying the set of roots where $|r|\le 1$, otherwise it would be considered unstable.

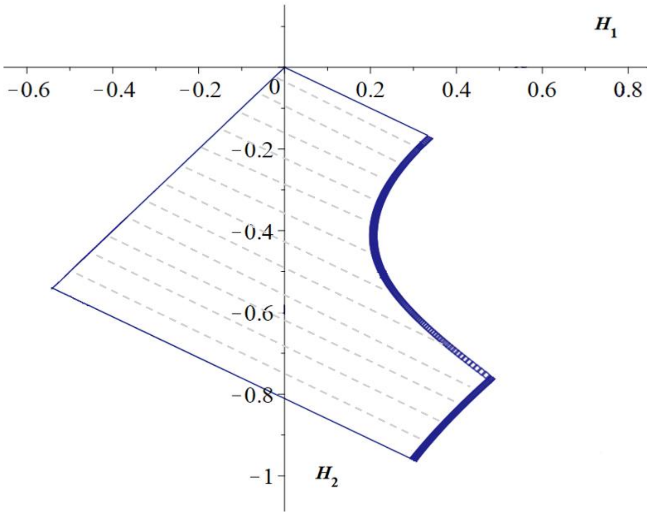

Figure 1: Stability region of DBM method.

RESULTS AND DISCUSSION

This manuscript uses the following abbreviations:

h	:	Step size.
TS	:	Total steps taken.
FCN	:	Function evaluations.
Time	:	Execution time taken in seconds
MAXE	:	Maximum absolute error.
RKS	:	Runge-Kutta method of order four with Simpson's $1/3$ rule.
ABM	:	Adam-Bashforth-Moulton three step method with composite Simpson's rule.
DMB	:	Diagonally multistep block method with composite Simpson's rule proposed in this study.

Several numerical problems are presented to evaluate the efficacy of the proposed approach.

Problem 1

Consider Zaidan (2012),

$$y'(t) = 1 - \frac{t^4}{3} + \int_0^t t\, u\, y(u-1)\, du,$$
$$\phi(t) = t + 1, \quad -1 \le t \le 0.$$

Theoretical solution: $y(t) = t + 1$ where $0 \le t \le 1$.

Table 1: Results of computations in solving Problem 1.

h	Method	MAXE	FCN	TS	Time
10^{-1}	RKS	8.133056e-03	40	10	0.023282
	ABM	3.316111e-03	27	10	0.023690
	DMB	1.884408e-06	11	7	0.010282
10^{-2}	RKS	1.546064e-02	400	100	0.074236
	ABM	4.058577e-04	297	100	0.025358
	DMB	1.875078e-11	101	52	0.014450
10^{-3}	RKS	1.654211e-02	4000	1000	0.595076
	ABM	4.155586e-05	2997	1000	0.038147
	DMB	4.440892e-16	1001	502	0.019876
10^{-4}	RKS	1.665417e-02	40000	10000	3.673743
	ABM	4.165556e-06	29997	10000	1.056073
	DMB	1.643130e-14	10001	5002	0.539982

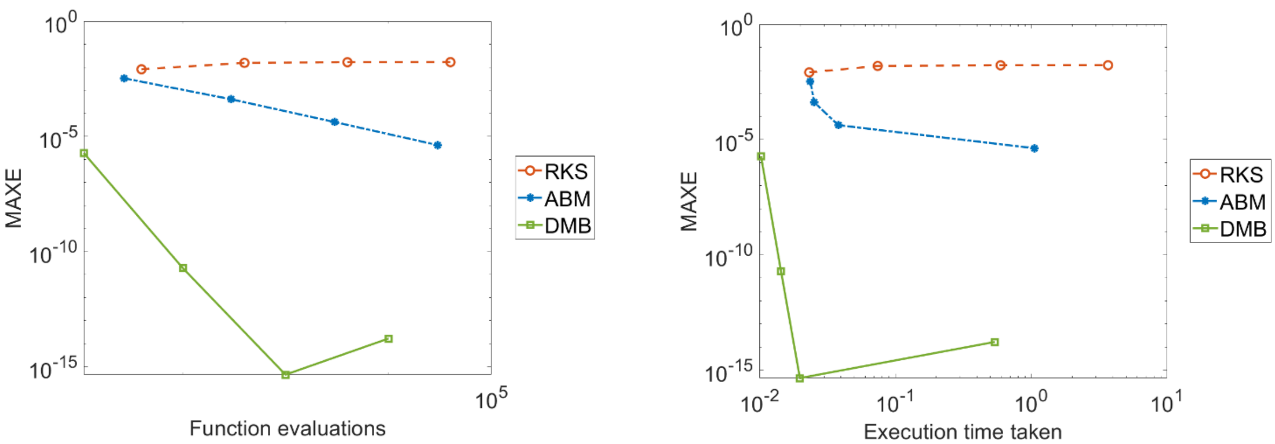

Figure 2: The efficiency curve for Problem 1.

Problem 2

Consider Salih et al. (2010),

$$y'(t) = 1 + t + t^2 - ty\left(t - \frac{1}{2}\right) + \int_0^t \exp(t-u)\, y\left(u - \frac{1}{2}\right) du,$$
$$\phi(t) = t + \frac{1}{2}, \quad -\frac{1}{2} \le t \le 0.$$

Theoretical solution: $y(t) = \exp(t) - \frac{1}{2}$ where $t \in \left[0, \frac{1}{2}\right]$.

Table 2: Results of computations in solving Problem 2.

h	**Method**	**MAXE**	**FCN**	**TS**	**Time**
5e-02	RKS	1.663650e-03	40	10	0.025499
	ABM	5.055619e-04	27	10	0.024292
	DMB	5.716653e-05	11	7	0.010994
5e-03	RKS	2.735074e-03	400	100	0.066722
	ABM	5.833119e-05	297	100	0.026718
	DMB	5.540669e-08	101	52	0.016108
5e-04	RKS	2.872346e-03	4000	1000	0.572643
	ABM	5.920416e-06	2997	1000	0.154679
	DMB	5.556938e-11	1001	502	0.075953
5e-05	RKS	2.886375e-03	40000	10000	8.292226
	ABM	5.929326e-07	29997	10000	12.133309
	DMB	4.807266e-14	10001	5002	6.018065

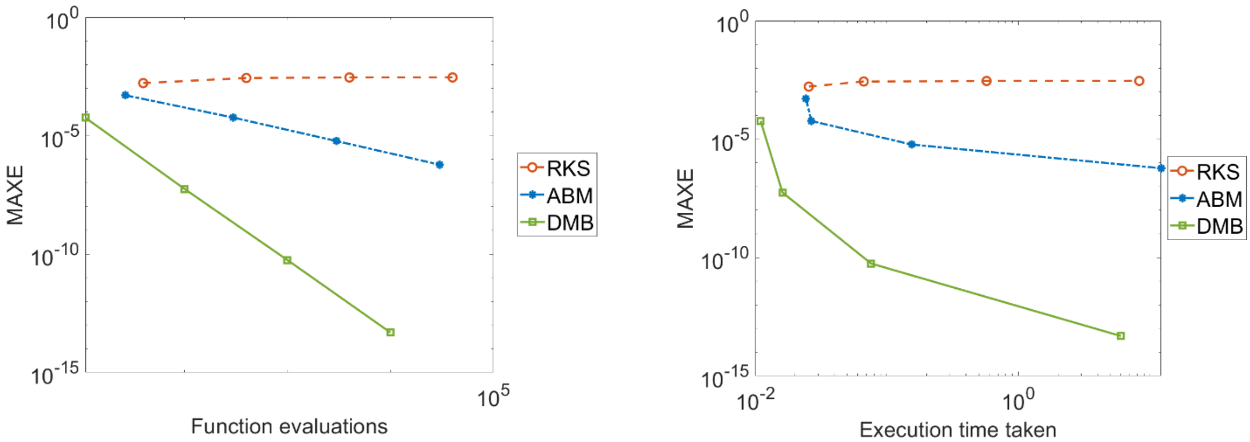

Figure 3: The efficiency curve for Problem 2.

Problem 3

Consider Salih et al. (2014)

$$y'(t) = \frac{1}{2}\big(1 - t + \exp(t)\big) - ty(t - \frac{1}{2}) + \int_0^t \exp(t-u)\, y\left(u - \frac{1}{2}\right) du,$$

$$\phi(t) = \exp(t) - \frac{1}{2}, \quad -\frac{1}{2} \le t \le 0,$$

Theoretical solution: $y(t) = t + \frac{1}{2}$ where $t \in \left[0, \frac{1}{2}\right]$.

Table 3: Results of computations in solving Problem 3.

h	**Method**	**MAXE**	**FCN**	**TS**	**Time**
5e-02	RKS	2.747701e-03	40	10	0.025013
	ABM	6.616172e-04	27	10	0.025648
	DMB	2.952084e-05	11	7	0.012541
5e-03	RKS	4.261864e-03	400	100	0.068041
	ABM	7.901441e-05	297	100	0.026290
	DMB	2.853821e-08	101	52	0.017451
5e-04	RKS	4.445272e-03	4000	1000	0.572060
	ABM	8.028347e-06	2997	1000	0.158862
	DMB	2.844180e-11	1001	502	0.084439
5e-05	RKS	4.463936e-03	40000	10000	9.917162
	ABM	8.041090e-07	29997	10000	12.646411
	DMB	3.663736e-14	10001	5002	6.084151

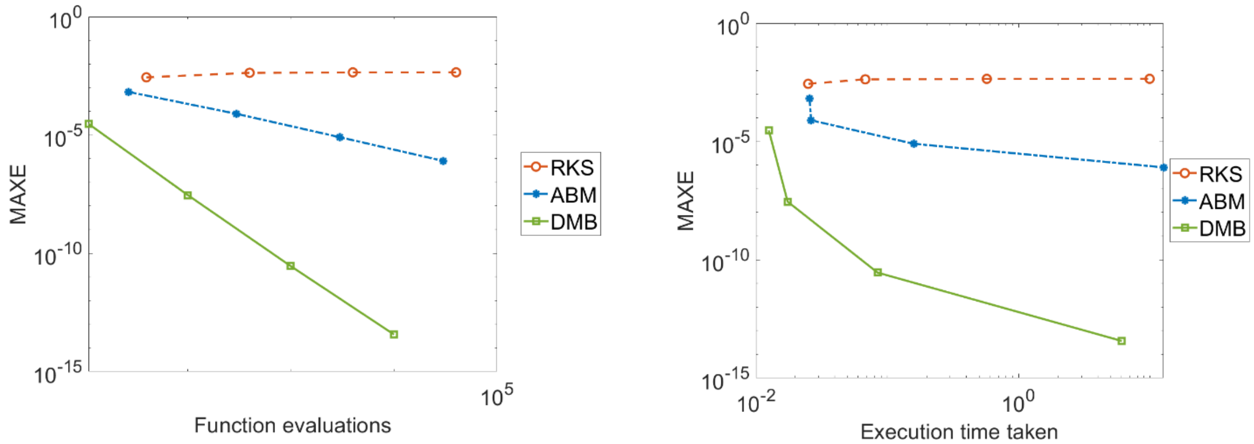

Figure 4: The efficiency curve for Problem 3.

Table 1 – 3 represent the computational performances of DMB, ABM and RKS in terms of accuracy, total function calls, total steps taken and execution time. It can be observed that the DMB achieved better performance compared to ABM and RKS. The proposed method, DMB, is inexpensive compared to the ABM and RKS in terms of total function evaluations. Therefore, the lesser function evaluations gave an advantage in terms of timing for DMB. The maximum error is lesser in DMB compared to ABM and RKS. The efficiency graphs of the numerical results can be referred in Figure 2 – 4.

CONCLUSION

The numerical results revealed that the advantage of DMB in obtaining better accuracy, faster in terms of the execution time, decreases the total number of function evaluations, and demonstrates better performance than the ABM and RKS. Therefore, the proposed method is recommended as an alternative well-performing iterative solver.

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