Menemui Matematik (Discovering Mathematics)
Vol. 45, No. 1: 76 - 84 (2023)

Direct Block Method for Solving Delay Differential Equations with Initial Conditions

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ABSTRACT

This study applied direct block multistep method with strategy to reduce function call in order to solve Delay Differential Equation (DDE) with initial condition. This study solves constant and pantograph delay type where the proposed method uses the previous calculated solutions and apply Lagrange interpolation method for the respective types of delay. The development of the proposed method is based on the diagonally implicit strategy where the interpolation points are defined diagonally to reduce the function calls so that it will reduce the computational time. Several numerical examples are taken into consideration to check the credibility of our strategies.

Keywords: Direct Block Method, Delay Differential Equations, Initial Value Problem

INTRODUCTION

A mathematical model recognised as Delay Differential Equations (DDE) has been employed to portray the real-world phenomena more precisely such as in control theory where there is delay during the amount of time required for a signal to arrive at the controlled object. In populations dynamic, delay also exist in breeding and maturation periods of species. Other than that, the biological process shows a delay during the period of cell division and cell production. In our daily lives, delays are so common that ignoring them is nearly impossible.

In mathematics, DDE are differential equations incorporating both past and present functions. The true or accurate solutions to the problems can be used to determine the solutions of DDE, however, not all DDE have true solutions. Alternatively, these solutions can be approximated by employing a variety of numerical techniques. We choose iterative method for this study but having high computational cost in iterative method is very common, therefore our motivation is to minimise the number of function calls generated by our suggested methods so that it able to lessen the computational cost. The second order constant DDE is identified as shown below:

$$y''(x) = f\left(x, y(x), y(x-\tau), y'(x-\tau)\right), x \in [a,b] \tag{1}$$

with the initial condition given by:

$$y(a) = \alpha, y(b) = \beta, \tag{2}$$

and initial function for the delay term is given by:

$$y(x) = \phi(x), [-\tau, a] \tag{3}$$

where τ is referring to time delay and $\phi(x)$ is referring to initial function provided to calculate the delay solutions. Meanwhile, the second order pantograph DDE is given as:

$$y''(x) = f\left(x, y(x), y(qx), y'(qx)\right), x \in [a,b] \tag{4}$$

where $0 < q < 1$. The delay term qx always lies in the interval $[a,b]$, hence the initial function, $\varphi(x)$, provided in the problems is not necessary.

The one step direct block method of two points and three points to solve second order constant DDE have been studied by Rasdi et al. (2013). They used the initial function to approximate the delay solutions or used divided difference interpolation. However, one step methods required many function calls for every iterations since they only have one previous function evaluation. This becomes the motivation to study multistep method to reduce function calls as multistep method have multiple previous function evaluations.

Various approaches based on various multistep methods have been addressed to solve first order DDE, however, just too little attention has been paid in solving second order DDE directly. Seong and Majid (2013) developed direct one-point diagonally Adams Moulton method of order 4 with the predictor of order three to solve second order constant DDE. Euler method was applied to compute the two starting initial values. The delay solution taken from the initial function given was stored to use in the future. They achieved better accuracy compared with the previous cubic spline method and comparable accuracy with the previous variable multistep method (VMM). Then, Seong et al. (2013) extended their previous studies to direct Adams Moulton method of order 5 with predictor of order 4. The direct Adams-Bashforth method was implemented to compute the first three starting points. The delay solution was computed by using the initial function given or reused the previous stored solutions.

One year later, direct two point predictor corrector block method of order 4 and 5 was introduced by Seong and Majid (2017) as an extension of their previous one point method. The first and second points were derived by using the same interpolating points and the Adams Moulton type of the corrector is said to be proposed in a fully implicit manner. The codes required two initial values and three initial values by Adams Bashforth method for the proposed method of order 4 and order 5, respectively. The execution at the delay term was similar as their previous works where no interpolation was needed to approximate the constant delay solution.

The above reviews are on the constant step size and constant order multistep method strategy. Isa and Ishak (2018) presented direct predictor-corrector of variable step size and variable order (VSVO) method. The backward difference formula (BDF) was used to generate the predictor and corrector formulas, which were subsequently expressed in divided difference form to cater the stiff problems. The method was applied to solve special second order stiff DDE. The delay solutions were solved by either using the initial function given, extending the grid points to use interpolation of predictor-corrector method, or used extrapolation of predictor-corrector method if the delay argument falls in the current interval.

This paper aims to enhance the research from Seong and Majid (2017) which is direct two point predictor corrector block method of order 4 and order 5. Our proposed method uses different interpolating points to derive the corrector formula in a diagonally implicit manner compared to fully implicit manner in their work. The reason behind this is to reduce the function calls for every iteration hence perhaps to reduce the total computational cost.

METHODOLOGY

The idea for derivation of direct multistep block methods is motivated from Majid and Suleiman (2007). Assume that first point, y_{i+1} and second point, y_{i+2} are the approximate solutions to the second order ODEs. The first and second points can be obtained by integrating both sides of second order ODEs over the interval $[x_i, x_{i+1}]$ and $[x_i, x_{i+2}]$ respectively. The integration is done twice where the first and second integration is for approximating y' and y solutions respectively.

First point, y_{i+1}:

$$\int_{x_i}^{x_{i+1}} y''(x)dx = \int_{x_i}^{x_{i+1}} f\left(x, y(x), y'(x)\right)dx$$

$$y'(x_{i+1}) = y'(x_i) + \int_{x_i}^{x_{i+1}} f\left(x, y(x), y'(x)\right)dx$$

and

$$\int_{x_i}^{x_{i+1}}\int_{x_i}^{x} y''(x)dx = \int_{x_i}^{x_{i+1}}\int_{x_i}^{x} f\left(x, y(x), y'(x)\right)dxdx$$

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \int_{x_i}^{x_{i+1}} \left(x_{i+1} - x\right) f\left(x, y(x), y'(x)\right)dx$$

Second point, y_{i+2}:

$$\int_{x_i}^{x_{i+2}} y''(x)dx = \int_{x_i}^{x_{i+2}} f\left(x, y(x), y'(x)\right)dx$$

$$y'(x_{i+2}) = y'(x_i) + \int_{x_i}^{x_{i+2}} f\left(x, y(x), y'(x)\right)dx$$

and

$$\int_{x_i}^{x_{i+2}}\int_{x_i}^{x} y''(x)dx = \int_{x_i}^{x_{i+2}}\int_{x_i}^{x} f\left(x, y(x), y'(x)\right)dxdx$$

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \int_{x_i}^{x_{i+2}} \left(x_{i+2} - x\right) f\left(x, y(x), y'(x)\right)dx$$

We then estimate the function *f(x,y,y')* with the Lagrange interpolating polynomial. Next, we going to derive our proposed methods based on order four and order five methods.

Order four (2PBM4):

The corrector of the order 4 of two-point diagonally direct block method (2PBM4) needs four points to interpolate in the Lagrange interpolating polynomial. These points are taken to be in diagonally implicit manner for y_{i+1} and y_{i+2} respectively as shown below:

$$\left\{\left(x_{i+1}, f_{i+1}\right), \left(x_i, f_i\right), \left(x_{i-1}, f_{i-1}\right), \left(x_{i-2}, f_{i-2}\right)\right\},$$
$$\left\{\left(x_{i+2}, f_{i+2}\right), \left(x_{i+1}, f_{i+1}\right), \left(x_i, f_i\right), \left(x_{i-1}, f_{i-1}\right)\right\}.$$

By considering $x = x_{i+1} + sh$, replacing $dx = hds$, the corrector formula finally obtained as follows:

$$y'_{i+1} = y'_i + \frac{h}{24}\left(9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}\right)$$
$$y_{i+1} = y_i + hy'_i + \frac{h^2}{360}\left(38f_{i+1} + 171f_i - 36f_{i-1} + 7f_{i-2}\right)$$
$$y'_{i+2} = y'_i + \frac{h}{3}\left(f_{i+2} + 4f_{i+1} + f_i\right)$$
$$y_{i+2} = y_i + 2hy'_i + \frac{h^2}{45}\left(2f_{i+2} + 54f_{i+1} + 36f_i - 2f_{i-1}\right)$$

(5)

Next, the predictor formula which is one order less than the corrector is obtained by using the same procedure but with different interpolating points. Both first and second points use the same interpolating points as follow:

$$\left\{(x_i\ , f_i),(x_{i-1}, f_{i-1}),(x_{i-2}, f_{i-2})\right\}.$$

Hence, the predictor formula finally derived as follows:

$$y'_{i+1} = y'_i + \frac{h}{12}\left(23f_i - 16f_{i-1} + 5f_{i-2}\right)$$
$$y_{i+1} = y_i + hy'_i + \frac{h^2}{24}\left(19f_i - 10f_{i-1} + 3f_{i-2}\right)$$
$$y'_{i+2} = y'_i + \frac{h}{3}\left(19f_i - 20f_{i-1} + 7f_{i-2}\right)$$
$$y_{i+2} = y_i + 2hy'_i + \frac{h^2}{3}\left(14f_i - 12f_{i-1} + 4f_{i-2}\right)$$

(6)

Order five (2PBM5):

By using the same procedure for deriving 2PBM4, the same technique is done to derive two points diagonally direct block multistep method of order 5 (2PBM5) but with different interpolating points such as below:

$$\left\{(x_{i+1}, f_{i+1}),(x_i\ , f_i),(x_{i-1}, f_{i-1}),(x_{i-2}, f_{i-2}),(x_{i-3}, f_{i-3})\right\},$$
$$\left\{(x_{i+2}, f_{i+2}),(x_{i+1}, f_{i+1}),(x_i\ , f_i),(x_{i-1}, f_{i-1}),(x_{i-2}, f_{i-2})\right\},$$

for first point, y_{i+1} and second point, y_{i+2} respectively. Therefore, the 2PBM5 corrector formula is shown below:

$$y'_{i+1} = y'_i + \frac{h}{720}\left(251f_{i+1} + 646f_i - 264f_{i-1} + 106f_{i-2} - 19f_{i-3}\right)$$
$$y_{i+1} = y_i + hy'_i + \frac{h^2}{1440}\left(135f_{i+1} + 752f_i - 246f_{i-1} + 96f_{i-2} - 17f_{i-3}\right)$$
$$y'_{i+2} = y'_i + \frac{h}{90}\left(29f_{i+2} + 124f_{i+1} + 24f_i + 4f_{i-1} - f_{i-2}\right)$$
$$y_{i+2} = y_i + 2hy'_i + \frac{h^2}{90}\left(5f_{i+2} + 104f_{i+1} + 78f_i - 8f_{i-1} + f_{i-2}\right)$$

(7)

Meanwhile, the predictor is order 4 and four points are needed to interpolate for both y_{i+1} and y_{i+2} as the following:

$\{(x_i\ ,\ f_i),(x_{i-1},\ f_{i-1}),(x_{i-2},\ f_{i-2}),(x_{i-3},\ f_{i-3})\}.$

The 2PBM5 predictor formula is then given by:

$$
\begin{aligned}
y'_{i+1} &= y'_i + \frac{h}{24}\left(55 f_i - 59 f_{i-1} + 37 f_{i-2} - 9 f_{i-3}\right) \\
y_{i+1} &= y_i + hy'_i + \frac{h^2}{360}\left(323 f_i - 264 f_{i-1} + 159 f_{i-2} - 38 f_{i-3}\right) \\
y'_{i+2} &= y'_i + \frac{h}{3}\left(27 f_i - 44 f_{i-1} + 31 f_{i-2} - 8 f_{i-3}\right) \\
y_{i+2} &= y_i + 2hy'_i + \frac{h^2}{45}\left(272 f_i - 366 f_{i-1} + 246 f_{i-2} - 62 f_{i-3}\right)
\end{aligned}
\tag{8}
$$

Below is the algorithm for applying the 2PBM4 and 2PBM5 to compute the solution for second order DDE:

Step 1: Define the step size $h = \frac{b-a}{N}$ and define all the initial values needed.

Step 2: Calculate the starting values using Euler's method and Modified Euler's method as predictor and corrector respectively.

Step 3: Calculate the delay argument value exist in the equation.

Step 4: Locate the position of delay arguments:

- For Constant Delay: If delay argument is less than initial x value, use the initial function given to approximate the delay solutions or else use the previous approximate delay solutions because x will be falling in the previous x value.
- For Pantograph Delay: If delay argument is falling in the previous x value, use the previous approximate delay solutions or else approximate the delay solutions by using Lagrange interpolation.

Step 5: Calculate the consecutive values by using the established method, 2PBM4 or 2PBM5.

NUMERICAL EXPERIMENTS AND COMPARISONS

There are two numerical problems for each second order constant and pantograph DDE type to be tested by applying both proposed methods, 2PBM4 and 2PBM5. Results obtained are compared with previous methods which are multistep and one-step method to observe the accuracy and the total function calls. Before we observe the results, below is the notations used in the tables:

h	: step size used.
MAXE	: maximum absolute errors (exact solution – approximate solution).
FCN	: total function calls for every iteration.
2PBM4	: two-points block diagonally implicit method order four proposed in this paper.
2PBM5	: two-points block diagonally implicit method order five proposed in this paper.
2PBFM4	: two-points block fully implicit multistep method order four proposed by Seong and Majid (2017).
2PBFM5	: two-points block fully implicit multistep method order five proposed by Seong and Majid (2017).
2PBDDE	: two-points block one step method proposed by Rasdi et al. (2013).
3PBDDE	: three-points block one step method proposed by Rasdi et al. (2013).

Second order constant DDE:

Problem 1:

$y''(x) = -\frac{1}{2}y(x) + \frac{1}{2}y(x-\pi), x \in [0,\pi]$

$y(x) = 1 - \sin(x), x \in [-\pi, 0]$

$y(0) = 1, y'(0) = -1$

Exact solution: $y(x) = 1 - \sin(x)$

Source: Rasdi et al. (2013).

Table 1: Numerical results for *Problem 1*.

h	METHOD	MAXE	FCN
$\frac{\pi}{30}$	**2PBM4**	**1.5018E-05**	**31**
	2PBFM4	1.5085E-05	58
	2PBM5	**7.5273E-07**	**53**
	2PBFM5	2.9388E-06	63
	2PBDDE	1.2110E-04	75
	3PBDDE	2.4141E-04	70
$\frac{\pi}{300}$	**2PBM4**	**1.3566E-09**	**301**
	2PBFM4	1.3070E-09	463
	2PBM5	**9.4480E-11**	**323**
	2PBFM5	3.3778E-10	468
	2PBDDE	1.1738E-07	750
	3PBDDE	2.3371E-07	700
$\frac{\pi}{3000}$	**2PBM4**	**1.3348E-13**	**3001**
	2PBFM4	5.5028E-12	4513
	2PBM5	**9.2149E-15**	**3023**
	2PBFM5	5.4137E-12	4518
	2PBDDE	9.7985E-11	7500
	3PBDDE	1.7811E-11	7000

Problem 2:

$y''(x) = y(x-\pi), x \in [0,\pi]$

$y(x) = \sin(x), x \in [-\pi, 0]$

$y(0) = 0, y'(0) = 1$

Exact solution: $y(x) = \sin(x)$

Source: Rasdi et al. (2013).

Table 2: Numerical results for *Problem 2*.

h	METHOD	MAXE	FCN
$\frac{\pi}{30}$	**2PBM4**	**5.8321E-05**	**31**
	2PBFM4	5.8321E-05	58
	2PBM5	**1.9614E-06**	**53**
	2PBFM5	8.1362E-06	63
	2PBDDE	1.9690E-06	75
	3PBDDE	2.9007E-06	70
$\frac{\pi}{300}$	**2PBM4**	**6.0629E-09**	**301**
	2PBFM4	6.0833E-09	463
	2PBM5	**2.1853E-10**	**323**
	2PBFM5	8.9857E-10	468
	2PBDDE	6.1050E-10	750
	3PBDDE	3.4812E-10	700
$\frac{\pi}{3000}$	**2PBM4**	**6.0902E-13**	**3001**
	2PBFM4	2.1246E-11	4513
	2PBM5	**2.0660E-14**	**3023**
	2PBM5	2.0722E-11	4518
	2PBDDE	8.2033E-10	7500
	3PBDDE	8.2030E-10	7000

Second order pantograph DDE:

Problem 3:

$$y''(x) = \frac{3}{4}y(x) + y\left(\frac{x}{2}\right) - x^2 + 2, x \in [0,1]$$

$$y(0) = 0, y'(0) = 0$$

Exact solution: $y(x) = x^2$

Source: Evans (2005).

Table 3: Numerical results for *Problem 3*.

h	METHOD	MAXE	FCN
0.1	**2PBM4**	**1.350E-05**	**7**
	2PBFM4	9.187E-04	16
	2PBM5	**9.230E-07**	**15**
	2PBFM5	4.312E-05	17
0.01	**2PBM4**	**1.593E-09**	**52**
	2PBFM4	2.612E-06	151
	2PBM5	**1.593E-09**	**102**
	2PBFM5	2.148E-06	152
0.001	**2PBM4**	**1.669E-13**	**502**
	2PBFM4	4.325E-08	1501
	2PBM5	**1.668e-013**	**1002**
	2PBFM5	1.099E-08	1502

Problem 4:

$$y''(x) = 1 - 2y^2\left(\frac{x}{2}\right), x \in [0,1]$$

$$y(0) = 0, y'(0) = 0$$

Exact solution: $y(x) = \cos(x)$

Source: Karakoc (2009).

Table 4: Numerical results for *Problem 4*.

h	METHOD	MAXE	FCN
0.1	**2PBM4**	**2.811E-05**	**9**
	2PBFM4	4.016E-03	16
	2PBM5	**4.492E-05**	**15**
	2PBFM5	8.978E-05	17
0.01	**2PBM4**	**1.169E-07**	**52**
	2PBFM4	7.352E-06	151
	2PBM5	**1.757E-07**	**102**
	2PBFM5	4.009E-06	152
0.001	**2PBM4**	**1.147E-10**	**502**
	2PBFM4	5.021E-08	1501
	2PBM5	**1.721E-10**	**1002**
	2PBFM5	2.256E-08	1502

DISCUSSION

Based on the numerical results, we have compare the approximate solutions of the proposed methods with the previous methods of direct block one step method in Rasdi et al. (2013) and direct block fully implicit multistep method in Seong and Majid (2017) in Problem 1 and Problem 2, but only compare with method from Seong and Majid (2017) in Problem 3 and Problem 4 since Rasdi et al. (2013) did not solve for pantograph delay. In Table 1, the proposed method of order 4, 2PBM4 give more accurate results compared to one step methods, 2PBDDE and 3PBDDE but is comparable compared to multistep method, 2PBFM4. The proposed method of order 5, 2PBM5 give the most accurate results compared to all the previous methods. Meanwhile, the total function calls, FCN for both proposed methods are lesser than all the previous methods, and much lesser than one step method. This is essential as our aim is to reduce the computational cost hence reducing the FCN as much as possible can reach this objective.

In Table 2, the proposed method of order 5, 2PBM5 perform better than order 4, 2PBM4 based on the accuracy. The accuracy of 2PBM5 is comparable with all the previous methods. Despite this, total function calls, FCN of both proposed methods are always lesser than all the previous methods.

In Table 3 and Table 4, both proposed methods perform well compared to previous multistep methods based on accuracy. The total function calls of our proposed methods also about 50% lesser than the previous fully implicit methods. The reason is due to our method is derived based on the diagonally form of the interpolation points while fully implicit is derived based on the fully implicit form where additional one function call is needed for every iteration of their first point.

CONCLUSION

In this study, we have shown and discussed the performance of the 2PBM4 and 2PBM5 to solve second order DDE of constant and pantograph delay type. Therefore, we can conclude that our proposed methods are suitable and capable to solve these problems especially in the way of saving the computational cost of the methods.

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