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Wang's Stretching/Shrinking Sheet Problem for Nanofluids with the Effects of Suction and Injection

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ABSTRACT
In this study, Wang's stretching/shrinking sheet problem for nanofluids with the effects of suction and injection is investigated. The non-linear partial differential equations are reduced to non-linear ordinary differential equations using similarity transformation. The transformed ordinary differential equations are then solved numerically by using bvp4c solver in MATLAB software. Three different types of nanoparticles, which are copper, alumina and titania (Cu, Al_2O_3, TiO_2) with water as the base fluid are considered and analyzed in this study. The effects of suction and injection, solid volume fraction and stretching/shrinking parameter on the fluid flow and heat transfer are evaluated. The numerical results are obtained for the velocity profiles, velocity g profiles and temperature profiles, as well as the skin friction coefficient and local Nusselt number are presented in the graphical form. The results show that suction improves the heat transfer of nanofluids. Dual solutions are found to exist for both suction and injection effects. For the shrinking case, dual solutions are also obtained, however unique solution found for the stretching case.

Keywords: Boundary Layer, Dual Solutions, Nanoparticles, Stretching/Shrinking, Suction/Injection

INTRODUCTION

Nanotechnology becomes the next level of study among researchers as it is set to be a future for the industrial revolution. Many researchers being involved in the study of nanotechnology due to its wide applications and its contribution towards engineering fields, medical fields and industrial processes. The term for nanofluids invented by Choi and Eastman (1995), who are pioneer in nanofluids work. Nanofluids refer to the mixed of nanoparticles which made from metals, oxides, carbides, or carbon nanotubes with some of the base fluids. Some examples of the base fluids are water, ethylene glycol and mineral oils. Since then, many studies involving nanofluids have been done where some of them are Xuan and Li (2000), Maïga et al. (2005), Kang et al. (2006), and many more. Xuan and Li (2000) studied the boiling heat transfer and natural convection of nanofluids, Maïga et al. (2005) studied the heat transfer of nanofluids in forced convection flow, while Kang et al. (2006) studied the thermal conductivity of nanofluids using experimental effective particle. Further, Abu-Nada (2008) discussed the applications of nanofluids for heat transfer enhancement of separated flows.

The nanoparticles which used in this study such as alumina, Al_2O_3 have been developed by Eastman et al. (1996) and Wang et al. (1999), while titania, TiO_2 prepared by Murshed et al. (2005) and copper, Cu proposed by Eastman et al. (1996). Not only that, there were also researchers involved in mathematical models of nanofluids, for instance, Bachok et al. (2012) who studied the flow and heat transfer characteristic of nanofluids on a moving flat plate, Junoh et al. (2019) revised the Buongiorno's nanofluids model with the effects of induced magnetic field and suction. Very recently, Kardri et al. (2022) solved the problem of MHD flow past a non-linear stretching or shrinking cylinder in nanofluids.

The study of nanofluids related to stretching and shrinking sheets done by many researchers due to its importance especially in engineering processes, namely plastic extrusion and tinning/coating processes and shrinking wrapping. Tiwari and Das (2007) proposed a model for solving the problem of stretching and shrinking sheets in nanofluids by studying the behaviour of nanofluids in a solid volume fraction. Later, Khan and Pop (2010) reported the model used in their study for the nanofluid which incorporates the effects of Brownian motion and thermophoresis.

Meanwhile, the pioneer study due to the shrinking sheet done by Miklavcic and Wang (2006), by considering viscous fluid flow. Later, Wang (2008) discovered that shrinking sheet consist of many unique characteristics by considering the shrinking sheet and stagnation point flow problems that pioneered by Hiemenz (1911). Further, Ali et al. (2013) solved numerically the solutions of Wang's stretching/shrinking sheet problem for nanofluids by extending the work by Wang (2008), using the model proposed by Tiwari and Das (2007). Ali et al. (2013) followed the governing equations given by Khanafer et al. (2003), where the problems of the heat transfer enhancement for utilizing nanofluids in various parameters in two-dimensional enclosure are studied. Not only that, Ali et al. (2013) also used the governing equations given by Oztop and Abu-Nada (2008) who studied the buoyancy forces that half heated enclosure to know how the characteristic of the heat transfer and the fluid flow using nanofluids. By using the model and the governing equations, Ali et al. (2013) found that there exist dual solutions for the shrinking sheet case. Due to non-unique solution obtained by Ali et al. (2013), Junoh et al. (2019) performed a stability analysis to determine which solution is stable and it was found that the first solution proposed by Ali et al. (2013) was physically stable compared to the second solution.

It is widely known that one of the mechanisms of stabilization of the fluid flows is by applying suction through the surface. In fluid dynamics, suction has relevance to many technological applications, namely, rotating machinery, ships and submarines with a particular significance for the laminar turbulent control of the aircraft wings by delaying the separation (Turkyilmazoglu, 2007). Motivated by the above-mentioned works, the aim of our study is to extend the work by Ali et al. (2013) with the effects of suction and injection towards Wang's stretching/shrinking sheet problem for nanofluids by considering three different types of nanofluids particles which are copper, alumina and titania (Cu, Al_2O_3, TiO_2).

MATHEMATICAL FORMULATIONS

Following Wang (2008) and Ali et al. (2013), the steady stagnation-point flow of a viscous and incompressible fluid over a continuously stretching or shrinking sheet in its own plane in water-based nanofluids is considered. It is assumed that the free stream velocity is $u_e(x) = ax$, where a is a positive constant, while the velocity of the stretching sheet is $u_x(x) = a(x+c)$, where $b > 0$ is the stretching rate and $b < 0$ is the shrinking rate and $-c$ is the location of the stretching origin. The *x*- axis is measured along the stretching surface and the *y*- axis is perpendicular to it. Under these assumptions, the basic governing equations of this problem can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho_{nf}}\frac{\partial p}{\partial x}+\frac{\mu_{nf}}{\rho_{nf}}\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho_{nf}}\frac{\partial p}{\partial y}+\frac{\mu_{nf}}{\rho_{nf}}\frac{\partial^2 v}{\partial y^2} \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_{nf}\frac{\partial^2 T}{\partial y^2} \tag{4}$$

where u and v are the velocity components along the x- axis and y- axis directions. The pressure is p, T is the nanofluids temperature, α_{nf} is the thermal diffusivity of the nanofluids, μ_{nf} and ρ_{nf} are the kinematic viscosity and the density of nanofluids, respectively.

Boundary conditions corresponding to the above Eqs. (1)-(4) are given as

$$\begin{aligned} &u=u_w(x)=b\,(x+c),\quad v=v_w(x),\quad T=T_w \quad \text{at} \quad y=0 \\ &u=u_e(x)=a\,x,\quad T=T_\infty \quad \text{as} \quad y\to\infty \end{aligned} \tag{5}$$

where u_e is the velocity at the edge of the boundary layer, v_w is the velocity at the condition of the surface, T_w is the temperature at the condition of the surface and T_∞ is the temperature of nanofluid at the condition outside the boundary layer. Some of the equations that have been employed by Khanafer et al. (2003) and Oztop and Abu-Nada (2008) are presented below:

$$\begin{aligned} &\mu_{nf}=\frac{\mu_f}{(1-\phi)^{2.5}},\quad \alpha_{nf}=\frac{k_{nf}}{\left(\rho C_p\right)_{nf}},\quad \rho_{nf}=(1-\phi)\rho_f+\phi\rho_s, \\ &\left(\rho C_p\right)_{nf}=(1-\phi)\left(\rho C_p\right)_f+\phi\left(\rho C_p\right)_s,\quad \frac{k_{nf}}{k_f}=\frac{\left(k_s+2k_f\right)-2\phi\left(k_f-k_s\right)}{\left(k_s+2k_f\right)+\phi\left(k_f-k_s\right)} \end{aligned} \tag{6}$$

where $\left(\rho C_p\right)_{nf}$ defines as the heat capacity of the nanofluids. While, k_{nf} defines the thermal conductivity of nanofluids, ϕ is the nanoparticle volume fraction, k_f and k_s are defined as the thermal conductivity of base fluid and solid, respectively. Equation (6) restricted to nanoparticles with spherical or near spherical (Ali et al., 2013).

Using the information from Wang et al. (2008) and Ali et al. (2013), we introduced the following similarity transformation:

$$\begin{aligned} &u=a\,x\,f'(\eta)+b\,c\,g(\eta),\ v=-\sqrt{a\nu_f}\,f(\eta), \\ &\eta=y\sqrt{a/\nu_f},\quad \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty} \end{aligned} \tag{7}$$

where primes denote differential with respect to η.

The expression p is obtained by using Eq. (3) and the boundary conditions (5) as follows:

$$p=p_0-\rho_{nf}\frac{a^2x^2}{2}-\mu_{nf}\frac{v^2}{2}+\mu_{nf}\frac{dv}{dy} \tag{8}$$

where p_0 is the stagnation pressure. Hence, we obtain:

$$-\frac{1}{\rho_{nf}}\frac{\partial p}{\partial x}=\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x} \quad \text{as} \quad y\to\infty . \tag{9}$$

By substituting Eqs. (7) and (9) into Eqs. (2)-(4), we obtain the following ordinary differential equations:

$$Af'''+ff''-f'^2+1=0 \tag{10}$$

$$Ag''+fg'-f'g+1=0 \tag{11}$$

$$B\frac{1}{\Pr}\theta''+f\theta'=0 \tag{12}$$

where the coefficients of A and B are defined as

$$A=\frac{1}{(1-\phi)^{2.5}\left(1-\phi+\frac{\phi\rho_s}{\rho_f}\right)},\; B=\frac{\frac{k_{nf}}{k_f}}{\left(1-\phi+\frac{\phi\left(\rho C_p\right)_s}{\left(\rho C_p\right)_f}\right)}.$$

The new boundary conditions become

$$\begin{aligned} &f(0)=S,\quad f'(0)=\varepsilon,\quad g(0)=1,\quad \theta(0)=1\\ &f'(\infty)=1,\quad g(\infty)=0,\quad \theta(\infty)=0 \end{aligned} \tag{13}$$

where

$$\Pr=\frac{\nu_f}{\alpha_f},\; \varepsilon=\frac{b}{a},\; S=\frac{-v_w(x)}{(a\nu_f)^{1/2}} \tag{14}$$

are the Prandtl number, the stretching/shrinking parameter and the suction/injection parameter, respectively. Here, $\varepsilon > 0$ is for stretching case and $\varepsilon < 0$ is for shrinking case. While, $S > 0$ refers to the suction effect and $S < 0$ refers to the injection effect. The physical quantities of interest in this study are presented by the skin friction coefficient C_f and the local Nusselt number Nu_x given below

$$C_f=\frac{\tau_w}{\rho_f u_e^2},\; Nu_x=\frac{xq_w}{k_f\left(T_w-T_\infty\right)} \tag{15}$$

where τ_w is the shear stress and q_w is surface heat flux that are given by

$$\tau_w=\mu_{nf}\left(\frac{\partial u}{\partial y}\right)_{y=0},\; q_w=k_f\left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{16}$$

Using the similarity variable (7), we obtain

$$C_f \, \mathrm{Re}_x^{1/2} = \frac{1}{(1-\phi)^{2.5}} \left[f''(0) + \frac{bc}{ax} g'(0) \right] \quad (17)$$

$$Nu_x \, \mathrm{Re}_x^{-1/2} = \left(\frac{k_{nf}}{k_f} \right) \theta'(0) \quad (18)$$

where $\mathrm{Re}_x = u_e x / \nu_f$ is the local Reynolds number.

RESULTS AND DISCUSSION

The transformed non-linear ordinary differential Eqs. (10)-(12) along with the new boundary conditions (13) have been solved numerically using the bvp4c method via MATLAB solver. In this study, we follow Oztop and Abu-Nada (2008) for the value of Prandtl number Pr which is 6.2 (water) and the value of nanoparticles volume fraction is $0 \le \phi \le 0.2$ in which $\phi = 0$ must be compatible with regular fluid. Each nanofluids has their own physical properties such as the specific heat capacity, density and thermal conductivity. Hence, following Oztop and Abu-Nada (2008), the thermophysical properties of the base fluid and nanofluids are stated in Table 1.

The accuracy of this method been validated by comparing the present results of the skin friction coefficient, $f''(0)$ with the values obtained from previous literatures with Wang (2008) and Junoh et al. (2019), as presented in Table 2. It is found that the results obtained show a favourable agreement. Therefore, we are confident that the numerical results obtained in this study are accurate.

Figures 1(a)-1(c) display the effect of the nanoparticle volume fraction, ϕ on the velocity, g and temperature profiles, respectively, for suction case $(S = 0.5)$. All Figs. 1(a)-1(c) exhibit dual profiles, therefore this admits the existence of dual solutions. When the value of the nanoparticle volume fraction, ϕ increases, the velocity profiles decrease for both solutions. The g profiles and temperature profiles show an increasing behaviour for the first solution, as the nanoparticle volume fraction increases. However, opposite effect occurs for the second solution. It can be seen clearly that all the profiles presented in the form of Figs. 1(a)-1(c) satisfy the boundary conditions (13) and produce asymptotic graphs. The boundary layer thickness for the second solution is larger compared to the first solution.

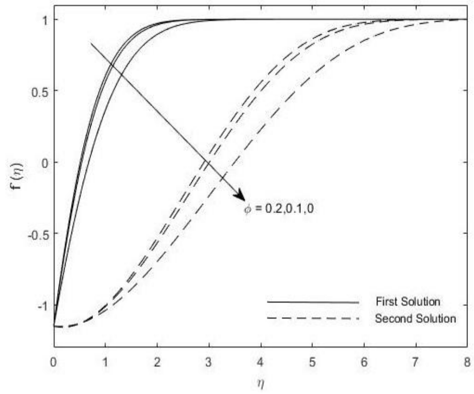
Table 1 Thermophysical properties of fluid and nanoparticles (Oztop and Abu-Nada, 2008)

Physical properties	**Fluid phase (water)**	***Cu***	***Al_2O_3***	***TiO_2***
C_p (J/kgK)	4179	385	765	686.2
ρ (kg/m^3)	997.1	8933	3970	4250
k (W/mk)	0.613	400	40	8.9538

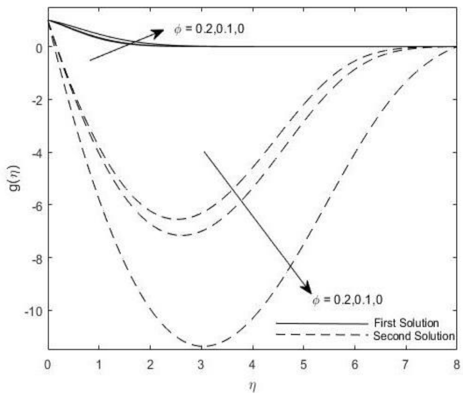
Table 2 Values of the skin friction coefficient $f''(0)$ for different ϕ and ε for *Cu*-water working fluid (when $S = 0$)

ε	Wang (2008)	Junoh et al. (2019)			Present		
	$\phi=0$	$\phi=0$	$\phi=0.1$	$\phi=0.2$	$\phi=0$	$\phi=0.1$	$\phi=0.2$
2	-1.88731	-1.887307	-2.217106	-2.298822	-1.887307	-2.217106	-2.298822
1	0	0	0	0	0	0	0
0.5	0.71330	0.713295	0.837940	0.868824	0.713295	0.837940	0.868824
0	1.232588	1.232588	1.447977	1.501346	1.232588	1.447977	1.501346
-0.5	1.49567	1.495669	1.757032	1.821791	1.495669	1.757032	1.821791
-1	1.32882	1.328817	1.561022	1.618557	1.328817	1.561022	1.618557
	[0]	[0]	[0]	[0]	[0]	[0]	[0]
-1.15	1.082231	1.082231	1.271347	1.318205	1.082231	1.271347	1.318205
	[0.116702]	[0.116701]	[0.137095]	[0.142148]	[0.116701]	[0.137095]	[0.142148]
-1.2		0.932473	1.095419	1.135793	0.932473	1.095419	1.135793
		[0.233649]	[0.274479]	[0.284596]	[0.233649]	[0.274479]	[0.284596]
-1.2465	0.55430	0.584282	0.686382	0.711680	0.584282	0.686382	0.711680
		[0.554295]	[0.651157]	[0.675157]	[0.554295]	[0.651157]	[0.675157]

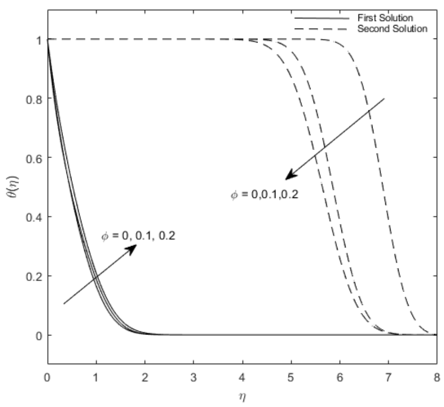
[ ] indicates second solution



(a)



(b)



(c)

FIGURE 1 (a) Velocity profile for various ϕ with suction (b) Velocity g profile for various ϕ with suction (c) Temperature profile for various ϕ with suction

Figures 2 and 3 show the skin friction coefficient, $f''(0)$ and the local Nusselt number, $-\theta'(0)$ for both suction case, namely $S = 0.5$ (black colour line) and injection case, namely $S = -0.5$ (blue colour line) when $\phi = 0.1$. It is found that unique solution exists for $\varepsilon > -1$. While, dual solutions are found to exist when $\varepsilon_c \le \varepsilon \le -1$, where ε_c is the critical point. The critical points for suction case as shown in Figs. 2 and 3 are $\varepsilon_c = -1.58, -1.525, -1.523$ for Cu, Al_2O_3 and TiO_2, respectively. No solution obtained for any values of $\varepsilon < \varepsilon_c$, where at this stage, the boundary layer separates from the surface. The critical values when injection is applied for all nanoparticles Cu, Al_2O_3 and TiO_2 are $\varepsilon_c = -1.05, -1.07, -1.065$. Thus, we can conclude that the range of solution becomes wider when suction is applied. This is due to suction can reduce drag at the surface, hence, delay the boundary layer separation. Notice that from Fig. 2 when $\varepsilon = 0$, there is no friction detected on the fluid-solid interface, $f''(0) = 0$. This phenomenon is due to the fluid move with the same velocity as the solid boundary.

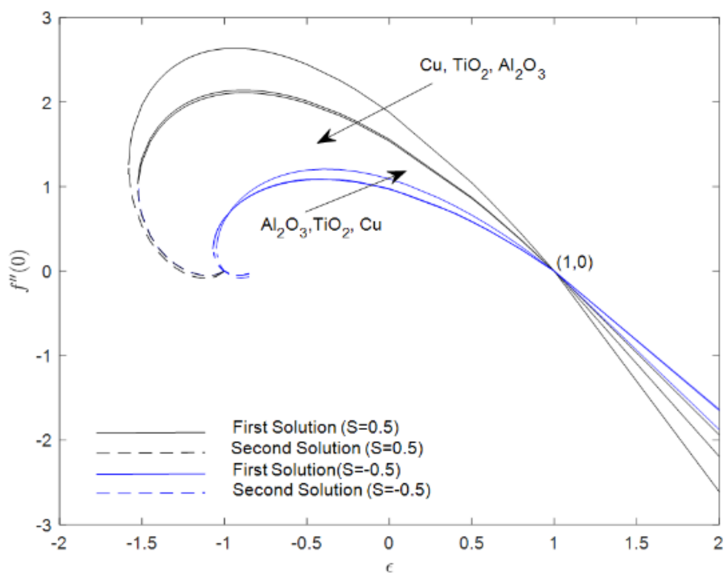


FIGURE 2 Variation of the skin friction coefficient for various nanoparticles with suction (S = 0.5) and injection (S = –0.5)

From Fig. 3, the local Nusselt number shows an increasing behaviour with ε. It is clearly shown that the local Nusselt number for Cu have a higher value compared to other nanoparticles, when the suction effect exists. This is due to the physical properties of fluid and nanoparticles where the thermal conductivity for Cu is greater than Al_2O_3 and TiO_2. However, for the selected nanoparticles in this study shows an opposite effect when injection is applied. For both suction and injection effects, increasing behaviour of the heat transfer rate at the surface is also observed for all nanoparticles. However, suction accelerates the heat transfer rate at the surface faster than injection. Both Figs. 2 and 3 also displayed the existence of dual solutions when the surface is shrunk.

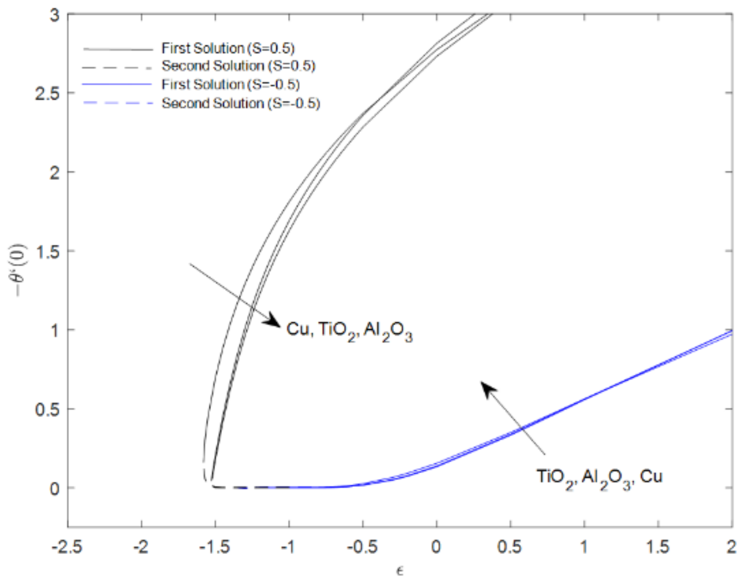


FIGURE 3 Variation of the local Nusselt number for various nanoparticles with suction (S = 0.5) and injection (S = –0.5)

CONCLUSION

In this study, the present problem is solved numerically by using bvp4c built in MATLAB software. The conclusions of the study are as follows:

- For all the profiles, the boundary layer thickness is larger for second solution.
- Dual solutions exist for both suction and injection cases.
- Dual solutions exist for shrinking case, however unique solution observed for stretching case.
- Suction effect widen the range of solutions. However, the range of solutions become smaller when injection is applied.

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