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On the Diophantine Equation $x^2 + 5^a47^b = y^n$

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ABSTRACT

Diophantine equation is known as a polynomial equation with two or more unknowns which only integral solutions are sought. This paper concentrates on finding an integral to the Diophantine equation $x^2 + 5^a \cdot 47^b = y^n$ for $0 < a, b \leq 30$ with (x, y) =1 and n =2. From this study, we found that there exists solution to the equation for $b < 4$.

Keywords: Diophantine equation, integral solution, exponential equation

INTRODUCTION

An equation with the restriction that only integer solutions are sought is called Diophantine equation. The main focus of this research is to solve exponential equation. Exponential Diophantine equation is an equation that has additional variable or variables occurring as exponents. Cohn (1993) stated that Diophantine equation in the form of $x^2 + C = y^n$ has been studied by Lebesque since 1850's where x, y are positive integers and also proved that there is no integral solution for $C = 1$. Then, Ljunggren (1943) solved the same equation for $C = 2$ and after that, Nagell (1955) solved for the value of $C = 3, 4$ and 5 as stated in Musa (2017). Cohn (1993) also studied the same equation for the 77 values of C between 1-100.

Bennett (2017) considered on the Diophantine equation $1 + 2^b + 6^b = y^q$ and solved for the exponential equation for $q = 3$ and partially for $q = 2$. The solution to the equation for a, b and y are positive integers are $(a, b, y) = (0,1,2)$ and $(9,3,9)$ if $q = 3$, and $(a, b, y) = (1,1,3)$ and $(3, 3, 15)$ for $q = 2$. Musa (2017) studied the Diophantine equation $x^2 + 5^a.p^b = y^n$ for $p = 29$. By using Lucas sequences, elliptic curves and MAGMA, they found that for the positive integers $a, b, x, y, n \geq 3$ with x and y are coprime, the solutions to the equation are $(x, y, a, b) = (2,9,2,1)$ and $(2, 3,2,1)$ for p =29 and $n = 3$ and 6 respectively.

Demirci (2017) was investigated all the solutions to the Diophantine equation $x^2 + 5^a p^b = y^n$ for p = 29, 41 for positive integers a, b, x, y, $n \geq 3$ with x and y relatively prime. From this research, the only solutions to the equation $x^2 + 5^a 29^b = y^n$, $x, y \geq 1$, gcd (x, y) = 1, $n \geq 3$, and $a, b \geq 0$ for $n = 3$ is (x, y, a, b) = (2, 9, 2, 1) and (x, y, a, b) = (2, 3, 2, 1) for $n = 6$. The only solutions of the equation $x^2 + 5^a 41^b = y^n$ where $x, y \geq 1$, gcd $(x, y) = 1$, $n \geq 3$, $a, b \geq 0$ for $n = 4$ is (x, y, a, b) = (840, 29, 0, 2) and (x, y, a, b) = (38, 5, 0, 2) for $n = 5$, and also (x, y, a, b) = (278, 5, 0, 2) for $n = 7$. Then, Alan and Zengin (2020) were studied on the Diophantine equation $x^2 + 3^a 41^b = y^n$ to determine all positive integer solutions (x, y, n, a, b) of the equation for positive integers a and b that integers x and y are relatively prime and $n \geq 3$. All

integer solutions are:
(x, y, n, a, b) = (280, 43, 3, 3, 1), (46, 13, 3, 4, 0), (10, 7, 3, 5, 0), (7030, 367, 3, 5, 1), (541160, 6643, 3, 11, 2), (16, 15, 4, 2, 1), (840, 29, 4, 0, 2), (38, 5, 5, 0, 2).

Borah and Dutta (2021) researched on the Diophantine equation $n^x + 24^y = z^2$ for $n \equiv 5$ or $7 \pmod 8$. They showed that it has a unique positive integral solution (2, 1, 7). Sudhanshu (2021) considered the exponential Diophantine equation $(2^{2m+1} - 1) + 13^n = z^2$ and discovered that there does not exist any whole number solution. It was mentioned that finding a generalized method for solving Diophantine equation is challenging, and in the case of the Diophantine equation $(2^{2m+1} - 1) + 13^n = z^2$, there were no solutions that was integer. This showed that some Diophantine equation might not have an integer solution.

In this paper, we consider the Diophantine equation $x^2 + 5^a 47^b = y^n$, for $0 < a, b \leq 30$, $(x, y) = 1$ and $n = 2$.

MAIN RESULT

In this study, the Diophantine equation that we consider is of the form $x^2 + 5^a 47^b = y^n$, for $0 < a, b \leq 30$, $n = 2$ with x and y are relatively prime. From the solution obtained, we could not construct the general form.

Theorem 1: Let a, b, x, y and n be positive integers. The integral solutions to the Diophantine equation $x^2 + 5^a\, 47^b = y^n$ for positive integers $a \leq 30$, $b = 1$, $(x, y) = 1$ and $n = 2$ are: (x, y, n, a, b) = (21,26,2,1,1), (117,118 2,1,1), (11,36,2,2,1), (587, 588,2,2,1), (39, 86,2,3,1), (2937, 2938, 2, 3, 1), (289, 336, 2, 4, 1), (14687,14688, 2,4,1), (1539, 1586, 2,5,1), (73437, 73438, 2, 5, 1), (7789, 7836, 2, 6, 1).

Proof:
In order to find an integral solution to the Diophantine equation $x^2 + 5^a\, 47^b = y^n$ for positive integers $a \leq 30$, $b = 1$, $(x, y) = 1$ and $n = 2$. We will consider six cases as follows:

Case 1: From the hypothesis, we begin with $a = 1$, $b = 1$, $(x, y) = 1$ and $n = 2$, we have

$$y^2 - x^2 = 5 \cdot 47 . \tag{1}$$

Factorize the above equation, we have

$$(y - x)(y + x) = 5 \cdot 47.$$

Since (5, 47) = 1, we have the following possibility,

$$(y - x) = 5,\ \ (y + x) = 47.$$

By solving these equations simultaneously, we have $(x, y) = (21, 26)$.

Now, in order to investigate other solutions of equation (1), we consider the parity of x and y. Since RHS is odd, then either one of x or y must be even.

Now, suppose x odd and y even. We let

$$x = 2k + 1 \text{ and } y = 2m, \tag{2}$$

then, equation (1) become,

$$(2m)^2 - (2k+1)^2 = 5 \cdot 47.$$

By expanding and simplifying the above equation, we have

$$m^2 - k^2 - k = 59.$$

That is,

$$m^2 = k(k+1) + 59.$$

Since LHS is a perfect square, then RHS also must be in a perfect square number in order to have a solution. Thus, we let $k + 1 = 59$. Therefore, we have

$$m^2 = k(k+1) + (k+1) = (k+1)^2.$$

Hence, we have $k = 58$ and $m = 59$. Then, substitute the value of m and k into equation (2), we will get $x = 117$ and $y = 118$.

Suppose x even and y odd. That is, $x = 2^\alpha m$ and $y = 2^\beta k + 1$ with $(2, m) = (2, k) = 1$ and $\alpha, \beta \geq 1$. Then, equation (1) become

$$(2^\beta k + 1)^2 - (2^\alpha m)^2 = 235. \tag{3}$$

By expanding and simplifying equation (3), we have

$$2^{2\beta}k^2 + 2^{\beta+1}k - 2^{2\alpha}m^2 = 2 \cdot 3^2 \cdot 13.$$

We will consider three cases by considering the possibilities of α, β as follows:

(i) Suppose $\alpha = \beta$. Then, equation (3) become

$$2^{\beta+1}(2^{\beta-1}k^2 + k - 2^{\beta-1}m^2) = 2 \cdot 3^2 \cdot 13.$$

Since $\beta \geq 1$, contradiction occurs. It is because RHS even and LHS is odd.

(ii) Suppose $\alpha \leq \beta$. Factorise equation (3), we obtain

$$2(2^{2\beta-1}k^2 + 2^\beta k - 2^{2\alpha-1}m^2) = 2 \cdot 3^2 \cdot 13.$$

Since $\alpha, \beta \geq 1$ and $\alpha \leq \beta$, contradiction occurs since LHS is even while RHS is odd.

(iii) Suppose $\beta \leq \alpha$. By using the same argument as in (ii), contradiction occurs since LHS is even while RHS is odd.

Case 2: Now, we consider for $a = 2$, and $b = 1$ then the equation (1) becomes,

$$x^2 + 5^2 \cdot 47 = y^2.$$

That is,

$$y^2 - x^2 = 5^2 \cdot 47 \tag{4}$$

$$(y - x)(y + x) = 5^2 \cdot 47.$$

Since $(5, 47) = 1$, suppose $(y - x) = 25$, $(y + x) = 47$. By solving these equations simultaneously, we have $x = 11$ and $y = 36$.

Now, we consider the parity of x and y. Suppose x odd and y even. That is,

$$x = 2k + 1 \text{ and } y = 2m. \tag{5}$$

From equation (4), substitute the value of x and y, we obtain

$$(2m)^2 - (2k+1)^2 = 11754.$$

By simplifying the equation, we have

$$m^2 = k(k+1) + 294\,.$$

Since LHS is a square number, then RHS also must be in a perfect number in order to have a solution. Thus, we let $k + 1 = 294$. Therefore, we have

$$m^2 = (k+1)^2.$$

Hence, $k = 293$ and $m = 294$. Then, substitute these values into equation (5), we obtain $x = 587$ and $y = 588$.

For the case x even and y odd, by using the similar argument as in Case 1, it can be proved that there does not exist solution to the equation for this case.

Case 3: Now, we consider $a = 3$, and $b = 1$ then the equation (1) becomes,

$$x^2 + 5^4 \cdot 47 = y^2.$$

That is,

$$(x-y)(x+y) = 5^3 \cdot 47.$$

Since $(5, 47) = 1$ and by using the argument as Case 2, we have $(x, y) = (39, 86)$ and $(x, y) = (2937, 2938)$.

Case 4: Suppose $a = 4$, then the equation (1) becomes,

$$y^2 - x^2 = 5^4 \cdot 47.$$

That is,

$$(x-y)(x+y) = 5^3 \cdot 47.$$

Since $(5, 47) = 1$ and by using the similar argument as in Case 1, we obtain $(x, y) = (289, 336)$ and $(x, y) = (14687, 14688)$.

Case 5: Suppose $a = 5$, then equation (1) becomes,

$$x^2 + 5^5 \cdot 47 = y^2.$$

That is,

$$(x-y)(x+y) = 5^5 \cdot 47.$$

Since $(5, 47) = 1$ and by using the same argument as in Case 1, we have two solutions $(x, y) = (1539{,}1586), (73437, 73438)$.

Case 6: Suppose $a = 6$, then the equation (1) becomes,

$$x^2 + 5^6 \cdot 47 = y^2.$$

That is,

$$y^2 - x^2 = 5^6 \cdot 47.$$

Since $(5, 47) = 1$, suppose $(y - x) = 47$, $(y + x) = 15625$. By solving these equations simultaneously, we have $(x,y) = (7789,7836)$. □

Conjecture 1: There has no positive integer solutions to the Diophantine equation $x^2 + 5^a \cdot 47^b = y^n$ where $6 < a \leq 30$, $b = 1$ for $(x, y) = 1$ and $n = 2$

The following theorem shows the form of positive integral solutions obtained for the Diophantine equation $x^2 + 5^a \cdot 47^b = y^n$ where positive integers $a \leq 30$, $b = 2$ and $(x, y)=1$ and $n = 2$.

Theorem 2: Suppose a, b, x, y and n be positive integers. The integer solutions to the Diophantine equation $x^2 + 5^a \cdot 47^b = y^n$ where $a \leq 30$, $b = 2$ for $(x, y) = 1$ and $n = 2$ are: (x, y, n, a, b) = (1102, 1107, 2, 1, 2), (5522, 5523, 2, 1, 2), (1092, 1117, 2, 2, 2), (27612, 27613, 2, 2, 2), (1042, 1167, 2, 3, 2), (792, 1417, 2, 4, 2), (458, 2667, 2, 5, 2).

Proof:
From equation (1), we will consider for the case, $b = 2$ for x and y are relatively prime and $n = 2$. Now, Suppose $a = 1$. Then the equation (1) becomes,

$$x^2 + 5 \cdot 47^2 = y^2. \tag{6}$$

From equation (6) we have

$$y^2 - x^2 = 5 \cdot 47^2. \tag{7}$$

Since RHS is odd then either one of x or y must be even. We will consider the following two cases.

Case 1: Suppose x even and y odd. That is, $x = 2^\alpha m$ and $y = 2^\beta k + 1$ with $(2, m) = (2, k) = 1$ and $\alpha, \beta \geq 1$. From (7) and by using the similar method as Case 1 in Theorem 1, the integral solution to the equation are $(x, y) = (1102, 1107), (5522, 5523)$.

Now, we will consider the parity of x and y in order to find other solutions to equation (7). Since RHS is odd, then either one of x or y must be even.

Suppose x odd and y even. That is, $x = 2^\alpha k + 1$ and $y = 2^\beta m$ with $(2, m) = (2,k)= 1$ and $\alpha, \beta \geq 1$. Then, equation (7) become

$$(2^\beta m)^2 - (2^\alpha k + 1)^2 = 5 \cdot 47^2. \tag{8}$$

By using the similar argument as Case 1 in Theorem 2, there does not exist integral solution to equation (8).

Case 2: Suppose $a = 2$, then the equation (7) becomes

$$x^2 + 5^2 \cdot 47^2 = y^2. \tag{9}$$

From equation (9), since RHS is odd then either one of x or y must be even. By using the similar argument as Case 1, the integral solution to the equation (9) are:

$$(x, y) = (1092, 1117),\ (27612, 27613).$$

Case 3: Suppose $a = 3$, then equation (7) becomes,

$$x^2 + 5^3 \cdot 47^2 = y^2. \tag{10}$$

From equation (10) we have

$$y^2 - x^2 = 5^3 \cdot 47^2. \tag{11}$$

Since RHS is odd then x and y must be in a different parity. By Thus, we will consider two cases as follows:

Case I: Suppose x even and y odd. That is $x = 2^{\alpha}k$ and $y = 2^{\beta}m + 1$ with $(2, m)$=(2,k)=1. By factorizing equation (11), we have

$$(y - x)(y + x) = 5^3 \cdot 47^2.$$

Since (5,47) = 1, suppose $(y - x) = 125$, $(y + x) = 2209$. By solving these equations simultaneously, we have the integral solution to the equation $(x, y) = (1042,1167)$.

Case II: Suppose x odd and y even. That is, $x = 2^{\alpha}k + 1$ and $y = 2^{\beta}m$ with $(2, m)$ =(2, k) = 1 and $\alpha, \beta \geq 1$. By using the similar argument as Case 1 in Theorem 2, there does not exists integral solution to equation (8).

Case 4: Suppose $a = 4$, then the equation (7) becomes,

$$x^2 + 5^4 \cdot 47^2 = y^2. \tag{12}$$

From equation (12), we have

$$y^2 - x^2 = 5^4 \cdot 47^2. \tag{13}$$

By factorizing equation (13), we have

$$(y - x)(y + x) = 5^4 \cdot 47^2.$$

Since (5, 47) = 1, suppose $(y - x) = 625$, $(y + x) = 2209$. By solving these equations simultaneously, we have $(x,y) = (792, 1417)$.

Now, we will consider the parity of x and y in order to find other solutions to equation (13). Since RHS is odd, then either one of x or y must be even.

Suppose x odd and y even. That is, $x = 2^{\alpha}k + 1$ and $y = 2^{\beta}m$ with $(2, m) = (2, k) = 1$ and $\alpha, \beta \geq 1$. By using the similar argument as Case 1 in Theorem 2, there does not exists integral solution to equation (13) .

Case 5: Suppose $a = 5$, then the equation (1) becomes,

$$x^2 + 5^5 \cdot 47^2 = y^2. \tag{14}$$

From equation (14), we have

$$y^2 - x^2 = 5^5 \cdot 47^2. \tag{15}$$

Since RHS is odd then either one of x or y must be even. By using the similar argument as Case 1, we have $(x,y) = (458, 2667)$.

By considering the parity of x and y to equation (15), since RHS is odd, then either one of x or y must be even.

Suppose x odd and y even. That is, $x = 2^{\alpha}k + 1$ and $y = 2^{\beta}m$ with $(2, m) = (2, k) =$ 1 and $\alpha, \beta \geq 1$. By using the similar argument as Case 1 in Theorem 2, there does not exist integral solution to equation (15). □

Conjecture 2 : There has no positive integer solutions to the Diophantine equation $x^2 + 5^a \cdot 47^b = y^n$ where $5 < a \leq 30$, $b = 2$ for $(x, y) = 1$ and $n = 2$.

The following theorem shows the form of positive integral solution obtained for the Diophantine equation $x^2 + 5^a \cdot 47^b = y^n$ where $a \leq 30$, $b = 3$ which $(x, y) = 1$ and $n = 2$.

Theorem 3: Suppose a, b, x, y and n be positive integers. The integral solutions to the Diophantine equation $x^2 + 5^a \cdot 47^b = y^n$ for $a \leq 30$, b=3, $(x, y) = 1$ and $n = 2$ is: (x, y, n, a, b) =(51909, 51914, 2, 1, 3).

Proof:

From the equation

$$x^2 + 5^a \cdot 47^3 = y^n, \tag{16}$$

we will consider two cases as follows:

Case 1: Suppose $a = 1$, then the equation (16) becomes,

$$x^2 + 5 \cdot 47^2 = y^2 \tag{17}$$

From equation (17), we have

$$y^2 - x^2 = 5 \cdot 47^2. \tag{18}$$

By using the same argument as Case 1 in Theorem 1, we obtain the integral solution to equation (18) is (x, y) = (51909, 51914).

By considering the parity of x and y to equation (18) and since RHS is odd, then either one of x or y must be even. Suppose x even and y odd. That is, $x = 2^\alpha k$ and $y = 2^\beta m + 1$ with $(2, m)$=$(2,k)$=1 and $\alpha, \beta \geq 1$. Then, substitute these values to (18) and by using the similar argument as Case 1 in Theorem 1, there does not exist solution to equation (18).

Conjecture 3: There has no positive integer solutions to the Diophantine equation $x^2 + 5^a \cdot 47^b = y^n$ where $1 < a \leq 30$, $b = 3$ for $(x, y) = 1$ and $n = 2$.

Conjecture 4 : There has no positive integer solutions to the Diophantine equation $x^2 + 5^a \cdot 47^b = y^n$ for $a \leq 30$, $3 < b \leq 30$, $(x, y) = 1$ and $n = 2$.

CONCLUSION

From this research, we found that the positive integral solutions (x, y, n, a, b) to the Diophantine equation $x^2 + 5^a \cdot 47^b = y^n$ for $a, b \leq 30$ which $(x, y) = 1$ and $n = 2$ are:

(x, y, n, a, b) =(21, 26, 2, 1, 1), (117, 118, 2, 1, 2), (11, 36, 2, 2, 1), (587, 588, 2, 2, 1), (39, 86, 2, 3, 1), (2937, 2938, 2, 3, 1), (289, 336, 2, 4, 1), (14687, 14688, 2, 4, 1), (1539, 1586, 2, 5, 1), (73437, 73438, 2, 5, 1), (7789, 7836, 2, 6, 1), (1102, 1107, 2, 1, 2), (5522, 5523, 2, 1, 2), (1092, 1117, 2, 2, 2),(27612, 27613, 2, 2, 2), (1042, 1167, 2, 3, 2), (792, 1417, 2, 4, 2),(458, 2667, 2, 5, 2) and (51909, 51914, 2, 1, 3). From the conjectures, we can conclude that there has no positive integer solutions of $x^2 + 5^a \cdot 47^b = y^n$ where $6 < a \leq 30$ for $b = 1$, $5 < a \leq 30$ for $b = 2$, $1 < a \leq 30$ for $b = 3$ while

$a \leq 30$ for $3 < b \leq 30$ with $(x, y) = 1$ and $n = 2$.

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