Menemui Matematik (Discovering Mathematics)
Vol. 44, No. 1: 28-34 (2022)

Bound on Some Diophantine Equation

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ABSTRACT

Diophantine equation is known as a polynomial equation with two or more unknowns which only integral solutions are sought. This paper will concentrate on finding the least upper bound to the Diophantine equation $x^2 + 2^a 7^b = y^n$ for $1 \leq a \leq 8$ and found that there exists an integral solution to the equation for $y = 2$ and $y > 2$ with the bound of n <15,000 and n < 11,000 respectively.

Keywords: Diophantine equation, exponential Diophantine equation, Baker's method and local method

INTRODUCTION

Diophantine equation is a polynomial equation with two or more unknowns for which only integral solutions are sought. Exponential Diophantine equation is an equation that has additional variables occurring as exponents. Cohn (1993) stated that Diophantine equation in the form of $x^2 + C = y^n$ has been studied by Lebesque since 1850's where x, y are positive integers and also proved that there is no integral solution for $C = 1$. Then, Ljunggren (1943) solved the same equation for $C = 2$ and after that, Nagell (1955) solved for the value of $C = 3, 4$ and 5 as stated in Musa (2017). Cohn (1993) also studied the same equation for the 77 values of C between 1-100. Deng (2015) studied the Diophantine equation of the form $x^2 + q^m = c^{2n}$ for $c > 1$ and $q^t + 1 = c^n$ where t is a positive integers and q an odd prime. The solution to these equations is $(x, m, n) = (c^2 - 1, t, 2,)$ by considering some cases where $m = t$. Meanwhile, Ismail et. al (2012) provided all solutions to the Diophantine equation $x^2 + 2^a \cdot 3^b \cdot 11^c = y^n$, for $n \geq 3$ by transforming the Diophantine equation into several elliptic equations written in cubic and quadratic models, and then determined the integral solutions.

Hajdu and Pink (2014) proved that there only three solutions to the Diophantine equation $1 + 2^a + x^b = y^n$. The solutions are (a, x ,b, y, n) = (4,43,3,282,2), (7,15,1,12,2) and (1,5,3,2,7). Certain condition are considered in this research where $0 < x < 50$ odd and $y^n > 100$ with a, b, y, n are positive integers. By using local argument combining with Baker's method, their results are proven. There are several steps to determine the integral solution to this diophantine equation. First, through local argument method, they show that a and n is small. If n is small, local argument are use to determine all solution. Baker's method is applied if a is small to bound n, then the remaining are solved by local methods and MAGMA as well. Berczes (2016) studied the Diophantine equation $1 + x^a + z^b = y^n$ and completely solved in a, b, x, y are positive integers, $n \geq 4$ for $z \leq 50$ and $x \equiv z (\bmod 2)$. Since, there is no pattern obtained in this research, then they used upper bound to solve this problem by finding upper sharp bound of n as the general formulae. For the case $a > 5$, they found that $n < 15{,}000$ if $y = 2$ and $n < 10{,}000$ if $y > 2$. For the case $1 \leq a \leq 5$, they obtained $n < 15,000$ if $y = 2$, has the only solutions $(x, a, z, b, y, n) = (22,1,45,2,2,11)$ and $n < 10{,}000$ if $y \neq 2$, there is no solution.

Bennett (2017) considered on the Diophantine equation $1 + 2^b + 6^b = y^q$ and solved for the exponential equation for $q = 3$ and partially for $q = 2$. The solution to the equation for a, b and y are positive integers are $(a, b, y) = (0,1,2)$ and $(9,3,9)$ if $q = 3$, and $(a, b, y) = (1,1,3)$ and $(3, 3, 15)$ for $q = 2$. Musa (2017) studied the Diophantine equation $x^2 + 5^a . p^b = y^n$ for $p = 29$. By using Lucas sequences, elliptic curves and MAGMA, they found that for the positive integers $a, b, x, y, n \geq 3$ with x and y are coprime, the solutions to the equation are $(x, y, a, b) = (2,9,2,1)$ and $(2, 3,2,1)$ for p =29 and $n = 3$ and 6 respectively. Then, Bakar et. al (2019) studied an integral solution to the Diophantine equation $5^x + p^m n^y = z^2$, by considering the equation x, m, n, y and z be positive integers and for $p > 5$ and p is prime. They found that the general solution to the is in the form of $(x,m,n,y,z) = (2r, t, p^t k^2, \pm 2k(5^r), 1, p^t k, \pm 5^r)$ for x even and $y = 1$, and $(x, m, n, y, z) = \left(2r, 2t, \frac{5^{2r-a}-5^a}{2p^t}, 2, \frac{5^{2r-a}+5^a}{2p^t}\right)$ for x even and $y = 2$.

In this paper, we consider the Diophantine equation $x^2 + 2^a . 7^b = y^n$, for $1 \leq a \leq 8$ and $n \geq 3$. This problem is extended from Yow et. al (2013) where the author only consider for n is even only.

MAIN RESULT

In this study, from the pattern of the solution obtained, we could not construct the general form. Thus, we find an upper bound for $3 \leq n \leq 15{,}000$. In order to solve this equation, we will use Baker's and local methods.

The Diophantine equation that we consider is of the form

$$x^2 + 2^a 7^b = y^n. \qquad (1)$$

Suppose $t = 2^a 7^b$. If $a \leq 8$ and $b \leq 7$, then we have

$$t = 2^a 7^b \leq 2^8 7^7 = 210827008 .$$

Then, (1) becomes

$$x^2 + t = y^n \qquad (2)$$

where y is a positive integer with $y \geq 2, n \geq 3$ and $t \leq 2^8 7^7$. Then, we used Baker's method of linear forms in logarithm of two algebraic numbers as follow,

$$h(\alpha) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log\left[1, \left|\alpha^{(i)}\right|\right]\right)$$

where a_0 is a leading coefficient of polynomial of α over Z. Now, we will consider the equation of the form

$$\Lambda = b_2 \log a_2 - b_1 \log a_1$$

where $\log \alpha_1$, $\log \alpha_2$, α_1 and α_2 are positive integers.
In order to get the bound we need results from the following three lemmas.

Lemma 1: Suppose x and t be positive integers with $1 \leq x \leq 50{,}000$ and $28 \leq t \leq 210827008$. Let, y and x be a solution of equation $x^2 + t = y^n$ with $n \geq 3$, then there exists an integral solution to the equation and the bound is

$$n < \begin{cases} 15{,}000 & if \quad y = 2 \\ 11{,}000 & if \quad y > 2 \end{cases}.$$

Proof:
By using a method used by Lajos and Pink (2014), for $n \geq 3$, we have the following equation

$$t = y^n - x^2 = (y - x^2)(y^{n-1} + y^{n-2}x^2 \dots\dots\dots + (x^2)^{n-1}). \quad (3)$$

Since $t = 2^8 7^b$, the equation (3) implies that $y - x^2 \geq 1$ which together with $y \geq 2$ and equation (3) gives

$$t > 2^{n-1}. \quad (4)$$

Since $a \leq 8$ and $b \leq 7$, we have $t \leq 2^8 7^7$ and substitute in (4), we obtain $n \leq 30$. Suppose $x^2 = x^b$ as in (2), we have $b = nB + r$, where $B \geq 0$ and $0 < r \leq n - 1$. Thus, by using local method, equation (3) becomes

$$\left| x^r \left(\frac{x^B}{y} \right)^n - 1 \right| > \frac{t}{y^n} \ . \quad (5)$$

Then, we set

$$\Lambda := r \log x - n \ \log \left(\frac{y}{x^B} \right). \quad (6)$$

By equation (5), we have $\Lambda \neq 0$. If $\left| x^r \left(\frac{x^B}{y} \right)^n - 1 \right| > \frac{1}{3}$, and $y \geq 2$.

Thus, (5) becomes

$$n < \frac{\log 3t}{\log 2}$$

where $t \leq 2^8 7^7$. Now, suppose

$$\left| x^r \left(\frac{x^B}{y} \right)^n - 1 \right| \leq \frac{1}{3}.$$

Hence, from equation (5) and (6), we have

$$|\Lambda| < \frac{2t}{y^n} \quad (7)$$

Now, we will derive a lower bound for $|\Lambda|$ that occur in equation (6). Since $x, t \in Z$, and from equation (2), we have $x \equiv y \pmod 2$. Then, we have

$$\frac{y}{x^B} = \alpha_1, \qquad x = a_2, \qquad n = b_1, \qquad r = b_2$$

By inequality $\left| x^r \left(\frac{x^B}{y} \right)^n - 1 \right| \leq \frac{1}{3}$, we obviously have $y > x^B$. Then, we choose

$$H(\alpha_1) = \begin{cases} H\left(\frac{y}{x^B} \right) \leq 1 & \text{if} \quad y = 2 \\ \left(\frac{y}{x^B} \right) \leq \log y & \text{if} \quad y > 2 \end{cases}$$

where $H(\alpha_2) = H(x) = \log x$. Since $r < n$ and $H(\alpha) \geq 1$, if $y \geq 2$.

$$h < \frac{n}{\log y} + n$$

Thus, we obtain the lower bound as follows

$$\log|\Lambda| > -25.2(\log x)H(\alpha_1) \max\left\{\log\left(\frac{n}{\log x} + n\right) + 0.38,10\right\}^2 \quad (8)$$

By comparing equations (7) and (8), we obtain

$$n < 25.2 \max\left\{\log\left(\frac{n}{\log x} + n\right) + 0.38,10\right\}^2 (\log x)\frac{H(\alpha_1)}{\log y} + \frac{\log 2t}{\log y} \quad (9)$$

where by $y \geq 2$, implies that either

$$n < 2520\,(\log x)\frac{H(\alpha_1)}{\log y} + \frac{\log 2t}{\log y} \quad (10)$$

or

$$n < 25.2\,\left\{\log\left(\frac{n}{\log x} + n\right) + 0.38\right\}^2 (\log x)\frac{H(\alpha_1)}{\log y} + \frac{\log 2t}{\log y}, \quad (11)$$

since

$$\frac{H(\alpha_1)}{\log y} = \begin{cases} \frac{1}{\log 2} & \text{if} \quad y = 2 \\ 1 & if \quad y > 2 \end{cases}.$$

Then, by equations (10) and (11) a simple calculation gives a bound for n valid for every $t \leq 2^8.7^7$ and $y \geq 2$. Thus, we conclude that if $y = 2$ and $y > 2$, then $n < 15{,}000$ and $n < 11{,}000$ respectively.

Lemma 2 Let a, b, x, y and n be positive integers. Then, there exists an integral solution to the Diophantine equation $x^2 + 2^a 7^b = y^n$ for $1 \leq a \leq 8$ for $y = 2$ and the bound is $n < 15{,}000$.

Proof:

In order to proof this lemma, we substitute the values of $1 \leq a \leq 8$ and $y = 2$ into the equation $x^2 + 2^a 7^b = y^n$, we obtain $(a, b, x, y, n) = (2,1,6,2,6)$, $(2,1,22,2,9)$, $(4,1,20,2,9)$ and $(6,1,8,2,9)$. By substituting these values, we will have t and $\frac{H(\alpha_1)}{\log y} = \frac{1}{\log 2}$ and substitute into (10), we have the following bound for n.

Suppose $(a, b, x, y, n) = (4,1,20,2,9)$, we have

$$n < 2520\,(\log 20)\frac{1}{\log 2} + \frac{\log 2(210{,}827{,}008)}{\log 2}.$$

Then, we obtain

$$n < 10{,}919.8$$

By using the same argument as above, we will obtain $n < 15{,}000$ for all cases of $(a, b, x, y, n) = (2,1,6,2,6)$, $(2,1,22,2,9)$ and $(6,1,8,2,9)$.

From all cases, there exist a solution (a, b, x, y, n) to the Diophantine equation $x^2 + 2^a 7^b = y^n$ with $1 \leq a \leq 8$, $y = 2$ and $n < 15{,}000$. Now, we consider for the case $y > 2$.

Lemma 3 Let a, b, x, y and n be positive integers, then there exists an integral solution to the Diophantine equation $x^2 + 2^a 7^b = y^n$ for $1 \leq a \leq 8$ for $y > 2$ and the bound is $n < 11{,}000$.

Proof:
By substitution the values of $1 \leq a \leq 8$ with $y > 2$, we have the following solution for the case $n = 3$ and $n = 6$

Table 1: Solution for $x^2 + 2^a 7^b = y^n$ for $y > 2$

n	a	b	x	y	Upper bound n
3	2	1	6	4	1989.5926
	2	1	22	8	3411.5566
	2	1	225	37	3980.5198
	2	5	134	44	5956.1514
	2	6	686	98	249.4647
	2	7	2058	196	8378.5338
	2	7	7546	392	9800.4978
	3	1	76	18	4768.4978
	4	1	20	8	3307.2470
	6	1	8	8	2304.4382
	6	1	1448	128	5338.8260
	6	2	4192	260	9157.1530
	6	3	104	32	5111.5754
	6	3	392	56	6563.7323
	6	3	15288	616	10573.2151
	6	3	1176	112	7766.0779
	6	7	2744	392	8693.37.94
	8	1	48	16	4265.3794
	8	1	176	32	5687.3434
	8	1	1800	148	8231.9381
	8	7	1072	176	7664.7423
6	2	7	2058	14	8378.5338
	8	1	48	4	4265.3794

From Table 1, suppose we choose the highest value of x, that is $(a, b, x, y, n) = (6,3,15288,616,3)$ and by using (10), we have

$$n < 2520\,(\log 15288)(1) + \frac{\log 2(210{,}827{,}008)}{\log 2}.$$

Then, we have $n < 10573.213$

Now, suppose we choose the smallest integers of x. That is, $(a, b, x, y, n) = (2,1,6,4,3)$ and from (10), we have

$$n < 2520\,(\log 6)(1) + \frac{\log 2(210{,}827{,}008)}{\log 2}.$$

Then, we have $n < 1989.592$.

From all cases, there exist a solution (a, b, x, y, n) to the Diophantine equation $x^2 + 2^a 7^b = y^n$ with $1 \le a \le 8$, $y = 2$ and $n < 11{,}000$ as asserted.

□

Theorem 1 Let a, b, x, y and n be positive integer. Then, there exists an integral solution to the Diophantine equation $x^2 + 2^a 7^b = y^n$ for $1 \le a \le 8$ and the bound is

$$n < \begin{cases} 15{,}000 & if \quad y = 2 \\ 11{,}000 & if \quad y > 2 \end{cases}$$

Proof:
Refer to Lemmas 1, 2 and 3.

CONCLUSION

From this study, we found that there exists an integral solution (*x*,*y*,*a*,*b*,*n*) to the Diophantine equation $x^2 + 2^a 7^b = y^n$ for $1 \le a \le 8$, $n \ge 3$ and $y \ge 2$. This study can be extended to find an integral solutions (*x*,*y*) for any value of *a*, *b* and *n* .

REFERENCES

Bakar, H. S., Sapar, S. H. and M. A. M. (2019). On the Diophantine Equation $5^x + p^m n^y = z^2$. *Malaysian Journal of Mathematical Sciences*, 13(S):41–50.

Bennett, M. A. (2017). The polynomial-exponential equation $1 + 2^a + 6^b = y^q$. *Period Math Hungary*, 75:387–397.

Berczes, A., H. L. M. T. and Pink, I. (2016). On the diophantine equation $1 + x^a + z^b = y^n$. *Journal of Combinatorics and Number* Theory, 8(2):145154.

Cohn J. H. E. (1993), The Diophantine equation $x^2 + C = y^n$, *Acta Arith.* 109**,** 205-206,

Deng M. J., "A note on the Diophantine equation $x^2 + q^m = C^{2n}$, *Proc. Japan Acad*., 91, (2015) Hajdu, L. and Pink, I. (2014), On the diophantine equation $1+2^a + x^b = y^n$. *Journal of Number Theory*, 143:1–13.

Ismail N. C., Musa D., Ilkar I., Luca F., Soydan G. (2012), On the Diophantine Equation $x^2 + 2^a \cdot 3^b \cdot 11^c = y^n$. (arXive 1021.0730v, Jan.)

Ljunggren W. (1943), About some arctangent equations that lead to interesting indefinite equations, *Ark. Mat. Astr.Fys.* 29A, 13 (1943), 1-11. German

Musa D. (2017), On the Diophantine Equation $x^2 + 5^a P^b = y^n$, Published by Faculty of Sciences and Mathematics, (31)

Nagell T. (1955), Contributions to the theory of a category of Diophantine equation of the second degree with two unknown, *Nova Acta Reg. Soc. Upsal IV Ser.16 Uppsala* , 1-38. French.

Yow, K.S., Sapar S. H and Atan, K. (2013). On the diophantine equation $x^2+4^a7^b = y^n$, *Pertanika Journal Science Technology*, 21(2):443–458.