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The Quotient Based Graphs for some Finite Non-Abelian Groups

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ABSTRACT

The quotient based graph of a group G with a normal subgroup H is introduced in this paper. The graph denoted by $\varphi_H(G)$ has its vertices as elements of G and the edge $e = \{x, y\}$ in G links x to y whenever $xy \in G - H = H^c$. The notion of complete normal subgroup $\mathscr{H}$ is introduced and we also showed that the complete bipartite graph $K_{n,n}$ is equicentric.

Keywords: Non-Abelian Groups, Complete Bipartite Graph, Complete Normal Subgroup, Eccentricity.

INTRODUCTION

There are a lot of works on graphs for groups and rings, and nature of graphs are determined by the conditions imposed on the vertices of the graphs as can be seen in [6]; where Anderson and Badawi worked on the total graph of a commutative ring R. Their graph is an undirected graph with two distinct elements x and y of the ring R been adjacent if and only if $x + y$ belongs to the zero divisor of R. As for Erfanian and Tolue in [7] they came up with the conjugate class graph of finite groups, where the graph has vertex set as the non-central elements of the group and two distinct vertices are adjacent if they are conjugate. Ahmad Abbasi's [4] work was on the T-graph of a commutative ring, in which he used a commutative ring with non-zero identity and a proper ideal I of R, where two vertices x and y in the graph are adjacent if and only if $x + y \in S(I)$ where

$$S(I) = \{a \in R : ra \in I, for\ some\ r \in R - I\}.$$

In this paper the quotient based graph of a group G relative to its normal subgroup H is investigated, where we considered some finite non-abelian groups. The groups are the dihedral groups, quaternion and the symmetric groups. The notations used are D_{2n} for the dihedral groups, Q_8 for quaternion and S_n for the symmetric groups of degree n. We also considered their normal subgroups both maximal and non-maximal. The subgroups of these non-abelian groups are normal if their index in the group is 2 [1] and the center $Z(G)$ of the groups are also normal as seen in [2]. The paper is structured into four sections; with the first being the introduction, in section two the quotient based graphs are considered using maximal subgroups while section three the graphs are studied using $Z(G)$ as the normal subgroups and in each case examples are given. Then lastly section four concludes the paper. The graphs in this paper are simple, undirected and connected graphs, for definitions and terminologies on graphs we refer to [3] and [8].

USING MAXIMAL NORMAL SUBGROUPS

***Definition* 2.1**. Quotient Based Graph: Let G be a finite non-trivial group and H a normal subgroup of G. A quotient based graph of G, relative to H is the graph $\boldsymbol{\varphi}_{\boldsymbol{H}(G)}$ of a group G in which x and y are adjacent whenever $\boldsymbol{xy} \in \boldsymbol{G}-\boldsymbol{H}$ for $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{G}$.

We then select the subgroups to use for the non-abelian groups under consideration.

* For the dihedral groups (D_{2n}):

where $D_{2n} = < a, b : a^n = b^2 = 1, bab^{-1} = a^{-1} >$

Let Let $H = < a : a^n = 1 >$, then H is cyclic and since the order of a group is the order of its generator then $|H| = n$. Applying the Lagrange's theorem shows that H is of index 2, hence it is normal according to [1] and also maximal since it's quotient group is simple [2]. So by definition the quotient based graph for D_{2n} relative to H would have a bipartite vertex set, H and L where $L = D_{2n} - H$.

* For the quaternion Q_8:

$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$

Let $H = < i : i^4 = 1 >$, then it is also true that H *is* cyclic and $|H| = 4$. Since $|Q_8| = 8$, this implies H is of index 2 and its quotient group is simple, so H is a maximal normal subgroup of Q_8. The vertex set for the graph is partitioned into two.

* For The symmetry group S_n:

Let $H = A_n$, i.e the alternating group. S_n is of order $2n$ whenever $|A_n| = n$. Hence, A_n is a maximal normal subgroup of S_n.

Theorem 2.2. *Let G be a finite non-abelian group with a maximal normal subgroup H such that the left cosets of G partition it into H and L then:*

i. $\forall\, x, y \in H\; xy \in H$
ii. $\forall\, x, y \in L\; xy \in H$
iii. $\forall\, x \in H$ *and* $\forall\, y \in L\; xy \in L$

Proof. The index of G by H is 2; the left cosets partition G into H and L where

$L = G - H$, *which means* $H \cap L = \emptyset$ *and* $H \cup L = G$.

i. H is a subgroup of G so $1 \in H$ and also satisfies the closure property, hence the result.

ii. L is not a subgroup of G but just a coset so $1 \notin L$ and does not satisfy the closure property, so $\forall\, x, y \in L\; xy \notin L$. since $H = G - L$ and $H \cup L = G$ then $xy \in H$.

iii. CASE 1: When $x = 1$ this means $xy = 1y = y \in L$.
CASE 2: When $x \neq 1$.
Note that $xy \neq 1$ since y is not the inverse of x and also not an element of H so $xy \notin H$ and hence belongs to L since $G - H = L$.

Theorem 2.3. *The quotient based graph for a non-abelian group G of order 2n with a maximal normal subgroup H of order n is a complete bipartite graph* $K_{n,n}$.

Proof:

Let G be a non-abelian group and $H \lhd G$, where H is a maximal subgroup. Is clear that the index of H in G is 2, i.e the left cosets partition G into two equal parts H and L *(where*$L = G - H$*) and this implies that* $|H| = |L| = n$. *This shows that the vertex set of the graph has been partitioned into two equal parts.* We then show the completeness of the graph; i.e $\forall\ x \in H$, *x* is adjacent to every element of L (say y) .

Theorem 2.2 shows that $\forall\ x \in H$, x is not adjacent to any element of H and $\forall\ y \in L$, y is not adjacent to any element of L. *But (iii) of the theorem shows that* $\forall\ x \in H$, *x is adjacent to each element of L and this completes the proof.*

Below are examples of the quotient based graphs for some groups.

Example 2.4. *The graph* $\varphi_{H(D_6)}$ is a complete bipartite graph $K_{3,3}$. Where $H = \{1, a, a^2\}$ and $L = \{b, ab, a^2b\}$, the adjacent vertices are***;***

$\{1, b\}, \{1, ab\}, \{1, a^2b\}, \{a, b\}, \{a, ab\}, \{a, a^2b\}, \{a^2, b\}, \{a^2, ab\}, \{a^2, a^2b\}$.

Example 2.5. The quotient based graph for Q_8 relative to $H = \{1, -1, i, -i\}$ is a complete bipartite graph $K_{4,4}$.

Example 2.6: When $H = A_4$ a maximal normal subgroup of S_4 is used to construct the quotient based graph $\varphi_{H(S_4)}$ we get a complete bipartite graph $K_{12,12}$.

We then look at some theorems that have to do with some properties of the bipartite graph;

Theorem 2.7. *A complete bipartite graph* $K_{n,n}$ *is equicentric and* $diam\left(K_{n,n}\right) = 2$.

Proof. The graph $K_{n,n}$ has two partite sets say A and B, since the graph is complete every vertex *x* in A is adjacent to every vertex *y* in B which means $d(x, y) = 1$. And it is clear that $\forall\ x, y$ in set A (or B) x and y are not adjacent but there is a path between x and y, with a vertex *w* of B (or A as the case may be) in between *x* and *y* resulting into a walk $\{x, w, y\}$; so $d(x, y) = 2$, indicating equal eccentricity.

Also observe that the path $\{x, w, y\}$ in $K_{n,n}$ indicates there is a diametrical path between every distinct vertices of A (or B). Thus for any vertex *v* in A (or B), $ecc_{max}(v) = 2 = diam\left(K_{n,n}\right)$.

Theorem 2.8. *The eccentric connectivity polynomial of a complete bipartite graph* $K_{n,n}$ *is a polynomial of degree 2 given by* $2n^2x^2$.

Proof. Theorem 2.7 has shown that the eccentricity $(v) = 2$ for every $v \in V(K_{n,n})$. And observe that for a complete bipartite graph $\deg(v) = n$ for every $v \in V(K_{n,n})$, we then *apply* the eccentric connectivity polynomial (E.C.P) formula as used in [5].

The polynomial is given by

$$E(K_{n,n},x) = \sum_{v \in V(K_{n,n})} \deg(v) x^{ecc(v)}$$

$$= \sum_{v \in V(K_{n,n})} nx^2$$

$$= \sum_{i}^{m} nx^2,$$

where $m = 2n$, the total number of vertices in $K_{n,n}$.

$$E(K_{n,n},x) = (nx^2)_1 + (nx^2)_2 + . \ . \ . (nx^2)_m$$

$$= 2n(nx^2) = 2n^2x^2$$

We now define the notion of the complete normal subgroup.

***Definition* 2.9**. Complete Normal Subgroup: A normal subgroup H of a group G is said to be complete if $\forall\ x \in H, x \neq e$ then x generates H, i.e $< x >= H$.

We denote the complete normal subgroup by $\mathcal{H}$.

Example 2.10

1. The subgroup $\mathcal{H} = < a: a^n = 1 >$ is a complete normal subgroup of D_{2n}, *for n prime.*
2. A_3 is complete in S_3.
3. The complete normal subgroups for the group of integers modulo n, (Z_n) are given by;
   i. For n odd:
      $\mathcal{H} = \{\mathrm{m}, 2\mathrm{m}, 3\mathrm{m}, \ldots, \mathrm{xm}\}$ where $x \neq 1$, x divides n and $m = n/x$.
   ii. For n even :
      - $\mathcal{H} = \{0,\ n/2\}$
      - and $\mathcal{H} = \{m, 2m, 3m, \ldots, xm\}$ when n=pq with p and q prime.

Note that when the normal subgroup is complete the quotient based graph is a bipartite graph.

QUOTIENT BASED GRAPH WITH $\boldsymbol{H = Z(G)}$

In this section we investigate the nature of $\varphi_{H(G)}$ when the center of the group is used as a normal subgroup.

It is a known fact that the center $Z(G)$ of a non-abelian group G is normal in G [2], $Z(G) = \{1\}$ for the symmetry group and the dihedral group when n is odd. $Z(G) = \{1, a^{n/2}\}$ for a dihedral group for even n and $Z(G) = \{1, i^2\}$ for the quaternion.

Theorem 3.1. The quotient based graph for a non-abelian group G with the normal subgroup $H = Z(G)$ *is a bipartite graph which is not complete.*

Proof:

Case 1: When $Z(G) = \{1\}$

i. In this case the set of vertices is partitioned into $H\ and\ G - \{1\}$, so for

$x \in H, x$ is adjacent to every $y \in L\ since\ xy = 1y = y \in L$.

$ii.$ $x \in L,\ x$ is adjacent to $y \in L$ except its inverse since $xy = 1 \in H$. This indicates the incompleteness since there exist $x, y \in L : xy = 1$.

Example 3.2. The $\varphi_{H(D_6)}$ for D_6 when $H = Z(G)$ is a bipartite graph with 6 vertices and 13 edges, where the adjacent vertices are;

$$\{1,a\},\{1,a^2\},\{1,b\},\{1,ab\},\{1,a^2b\},\{a,b\},\{a,ab\},\{a,a^2b\}$$

$$\{a^2,b\},\{a^2,ab\},\{a^2,a^2b\},\{b,ab\},\{b,a^2b\}$$

Example 3.3. The quotient based graph $\varphi_{H(Q_8)}$ for Q_8 relative to $H = Z(G) = \{1,-1\}$ is a graph with 8 vertices and 24 edges which is bipartite but not complete.

CONCLUSION

In this paper we investigated the quotient based graphs for non-abelian groups relative to their normal subgroups. The graphs are complete bipartite graphs $(K_{n,n})$ when maximal normal sub-groups are used, the eccentricity and eccentric connectivity polynomials for $K_{n,n}$ are generalized.

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