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Solution of heat equation with Galerkin weighted residual method in addition with RKACeM(4,4) method

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ABSTRACT

In this study, a one-dimensional heat equation, subject to non-homogeneous Dirichlet boundary conditions (BCs) was solved by the well-known and efficient Galerkin weighted residual method in coordination with the embedded RKACeM(4,4) technique, which is based on the Runge-Kutta arithmetic mean and centroid mean methods each of order 4. The approximate solution was assumed to be the linear combination of the finite number of basis functions chosen from the solution space satisfying the boundary conditions. Imposing Galerkin requirement on the residual of the approximate solution as well as on initial condition, a system of first order ordinary differential equations (ODEs) of initial valued was obtained. Embedded RKACeM(4,4) method was then employed to solve the resulting system of ODEs, where the solution of the system, in turn, provided the required approximate solution to the given problem. The obtained results compared well with the analytic solution of the problem and were found to be more suitable with regard to the root mean square error values over the results obtained with other sophisticated techniques.

Keywords: Heat equation, Dirichlet boundary conditions, Galerkin weighted residual method, RKACeM(4,4) method

INTRODUCTION

Partial differential equations (PDEs) play an important role in explaining natural phenomena and, therefore, many areas of science and engineering seek solutions to these equations. But it is found that a large number of PDEs do not have analytic solutions, then one may rely on an approximated solution (Tadmor, 2012). As in Petrolito (1998), the modern treatment of PDEs relies heavily on the approximation methods, which provide a way to find out approximate solutions to these PDEs. To get an approximate solution of a PDE numerically, it is handled with a specific discretization scheme, and three discretization schemes are finite difference, finite element, and finite volume. The efficient and foremost among the approximation methods is finite element method (FEM), which is the most powerful numerical approach to solve PDEs (Gockenbach, 2005). It has now-a-days become a standard method for analyzing different problems arising in the field of thermo-mechanics, solid mechanics, biomechanics, fluid mechanics and about every branch of physics (Sharma et al., 2011). FEM is developed based on the weak formulation of a problem, which approximates its exact solution. The accuracy of such a solution is subject to the number and size of the elements, and the types of functions that are considered within the elements (Liu and Glass, 2013). FEM works based on the principle of minimizing the error generated from conceding a trial solution. It is found that different types of FEMs depending on minimizing process of generated errors belong to a family of weighted residual methods (Lindgren, 2009). In weighted residual methods, a residual function of a given differential equation (DE) is obtained by substituting a trial solution into the given equation, where such a solution is a linear combination of linearly independent basis functions satisfying the BCs of the given problem (Finlayson, 1972). In Galerkin weighted residual method (GWRM), one of the powerful weighted residual methods, the residue is forced to be zero by making orthogonal to each member of the basis functions (McGrattan, 1997). Setting inner products of the residual and basis functions to zero, a system of equations is obtained, which on solving, in turn, yields an approximate solution to the given problem. However, for solving the PDEs with the aid of a numerical approach, it is important to distinguish the type of the PDEs, which can be classified as elliptic, parabolic and hyperbolic types. Elliptic equations usually arise from a physical problem that involves steady state cases. Hyperbolic PDEs customarily arise in relation to mechanical oscillators, such as a vibrating string, or in convection driven transport problems while a parabolic PDE can be found to arise in time dependent diffusion problems, for example, heat equation. It is the notion of the study to solve a parabolic type PDEs using GWRM with a sophisticated time integrator to have an efficient solution. As a test case, one dimensional heat equation (ODHE) with nonhomogeneous Dirichlet BCs is chosen. It is pertinent to mention here that heat is the energy, transferred from one body to another as a result of a difference in temperature, which describes the distribution of heat in a particular space over time. The equation refers to an irreversible process and makes a difference in temperature between the previous and next steps (Dabral et al., 2011). One-dimensional heat equation has been studied extensively by several authors adopting different approaches for its applications in science and engineering (see Ekolin, 1991; Liu, 1999; Sun and Zhang, 2003; Cagler et al., 2008; Mohebbi and Dehghan, 2010, Tatari and Dehghan, 2010). The FEM has great flexibility and importance in solving heat equation (El-Morsy and El-Azab, 2012; Kalyani and Rao, 2013). But in solving ODHE using GWRM, a system of first order ODEs is obtained, which can be solved with the aid of any sophisticated time integrator. At this juncture, it is suitable to note that RKACeM(4,4), an embedded technique, combined with two Runge-Kutta (RK) methods, namely RK based on the arithmetic mean (RKAM) and centriodal mean (RKCeM) each of order 4, is better over the RKAM method and RK Fehlberg (RKF(4,5)) method with respect to accuracy and central processing unit (CPU) time to solve an IVP or a system of IVPs (Murugesan et al., 2002). Thus, an accurate and efficient solution of ODHE can be achieved by using the GWRM in coordination with the RKACeM (4,4) technique. Accordingly, our current work attempts to implement the GWRM in coordination with the RKACeM(4,4) method for solving the specified equation with non-homogenous Dirichlet BCs. It is also our aim to show how the results emanated from this different approach support the results came out through some sophisticated techniques, namely, the finite difference method (FDM), the numerical method of lines (MOLs) in addition to RK(4,4) technique, the GWRM with the RK(4,4) method and analytical results. It is also seen the variation in residual curve obtained from the difference of the results calculated using the mentioned approaches with the analytic result. All the residual curves observed deeply and the residual calculated for the present study found to have smaller variations about zero. Actually, the study focused on the approach that can calculate a better result in very nearer to beginning time.

The rest of the part of the paper is arranged as follows. An overview of GWRM is presented in “A short note on GWRM” section. The model equation and boundary conditions are stated in “Problem statement” section. The integration procedure and error estimated process are discussed in detail in “Methodology” section and its subsections. Model emanated results, its discussion and validation of the model are presented in "Result and discussion" section. Finally, conclusion on the basis of obtained results is put forward in "Conclusion".

A SHORT NOTE ON GWRM

GWRM is an approximation method coined by Russian mathematicians Borish Grigoryevich Galerkin (1871–1945) for finding out an approximate solution of a continuous operator problem. The method converts the problem to a discrete one by making a weak formulation of the problem characterizing with a finite set of basis functions of the solution space to the problem. To find an approximate solution of a DE having the form

$$L[U(x)] = f(x), \tag{1}$$

on boundary

$$B[U] = [a, b], \tag{2}$$

where L is a differential operator and f is a given function, the GWRM introduces a trial solution $\tilde{u}(x)$ of Eq. (1) as

$$\tilde{u}(x) = \phi_o(x) + \sum_{j=1}^{n} c_j \phi_j(x), \tag{3}$$

where ϕ_j are basis functions and c_j are the coefficients to be determined.

The basis functions are linearly independent and satisfy the BCs of the problem. According to the principle of GWRM, the resulting values of c_j linearly combined with the basis functions give an approximate solution to the given problem through Eq. (3). This approximation method then introduces an error $R(x)$, called residual, which can be set as $R(x) = L(\tilde{u}) - f(x)$, and then attempt to make the residual to be zero relative to a weighting function W_i as

$$\langle W_i, R \rangle = 0, \tag{4}$$

which is known as Galerkin requirement.

In GWRM, weighting functions W_i are self-basis functions. Therefore, the Galerkin requirement takes the following form $\langle \phi_i, R \rangle = 0,$, which implies that

$$\int_a^b \phi_i R \, dx = 0,$$

or, $\int_a^b \phi_i L(\tilde{u}) \, dx = \int_a^b \phi_i f(x) \, dx,$

which represents a system of n algebraic equations or a system of n ODEs. Solving this system of equations, one can obtain the values of c_i and hence the approximate solution to the given problem.

PROBLEM STATEMENT

As in Ahmad and Yaacob (2013), ODHE is considered as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; 0 < x < 1; t > 0, \tag{5}$$

subject to initial condition (IC)

$$u(x, t = 0) = 70\ ^oC, \tag{6}$$

and the Dirichlet BCs

$$u(x = 0, t) = 50\ ^oC, \tag{7}$$

$$u(x = 1, t) = 20\ ^oC, \tag{8}$$

with the analytic solution obtained by using the method of separation of variables as

$$u(x,t) = 50 - 3x + 2\sum_{n=1}^{\infty}\left[\int_0^1 (20 + 30x)\sin\xi\ \ d\xi\right] e^{-n^2\pi^2 t}\sin(n\pi x), \tag{9}$$

where the symbols play the same meaning as stated in Ahmad and Yaacob (2013).

METHODOLOGY

Choice of basis functions

As a basis of the solution space of the stated problem represented by Eq. (5), a usual basis of polynomials $\phi_j = x^j - x^{j+1}$ is chosen. The linear combination of the linearly independent basis functions satisfying the BCs is considered as a trial solution. Thus, the trial solution for the chosen basis functions of the given problem specified by Eq. (5) can be represented as

$$\tilde{u}(x,t) = \phi_0(x) + \sum_{j=1}^{N} c_j(t)(x^j - x^{j+1}), \tag{10}$$

with $\phi_o(x) = z_a + \frac{z_b - z_a}{b-a}(x - a)$, whereas for the given problem, $z_a = 50, z_b = 20, a = 0\ and\ b = 1.$ i.e., $\phi_o(x) = 50 - 30x$.

Therefore, the trial solution gets the following form:

$$\tilde{u}(x,t) = 50 - 30x + \sum_{j=1}^{N} c_j(t)(x^j - x^{j+1}). \tag{11}$$

Now, the residual for the trial solution can be set to the form:

$$R = \sum_j^N \dot{c}_j(t)\left(x^j - x^{j+1}\right) - \sum_{j=1}^{N} c_j\left[j(j-1)x^{j-2} - j(j+1)x^{j-1}\right].$$

The Galerkin's requirement, $\langle\phi_i, R\rangle$, $i = 1, 2, 3, \dots, N$ then yields to form a system of first order ODEs with unknowns c_i as

$$\sum_{j=1}^{N} \dot{c}_j(t)\left(\frac{1}{i+j+1} - \frac{2}{i+j+2} + \frac{1}{i+j+3}\right) = \sum_{j=1}^{N} c_j(t)\left[\frac{j(j-1)}{i+j-1} - \frac{2j^2}{i+j} + \frac{j(j+1)}{i+j+1}\right]. \tag{12}$$

The solution of this system of ODEs gives the values of c_i while the ICs for solving the above system of equations are obtained by imposing Galerkin requirement on the residual of the IC.

Therefore, imposing Galerkin requirement on the residual of the IC, we have

$$\langle \phi_i, u(x,0) - \tilde{u}(x,0) \rangle,$$

which yields to have

$$\sum_{j=1}^{N} c_j(0) \left(\frac{1}{i+j+1} - \frac{2}{i+j+2} + \frac{1}{i+j+3} \right) = \left[\frac{20}{i+1} - \frac{10}{i+2} + \frac{30}{i+3} \right]. \quad (13)$$

Equation (13) provides a set of IC to solve the system of ODEs given by Eq. (12).

Integration procedure

Now to attain our solution of interest, the system of ODEs given by Eq. (12) are to be solved with the aid of the ICs specified by Eq. (13). To solve Eq. (12), as aforementioned, we adopted the embedded RKACeM(4,4) method, to attain an efficient solution. A gist of it is presented in the following sub section.

A short note on RKACeM(4,4) method

RKACeM(4,4) method is an adaptive method that incorporates the step size adapting the number and position of nodes ensuring that the truncation error was kept within a specified bound (Burden and Faires, 2010). This is a novel approach for solving IVPs with error control (EC) by formulating an embedded method involving RK method based on arithmetic mean (AM) and centroidal mean (CeM) methods (Murugesan et al., 2002).

We consider the IVP as under:

$$y' = f(x,y), a \leq x \leq b, \text{ with IC } y(a) = \alpha.$$

The approximation technique of the RKAM(4,4) method can be presented as

$$y_{n+1} = y_n + \frac{h}{3}\left[\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2} + \frac{k_3+k_4}{2}\right],$$

where

$$k_1 = f(x_n, y_n),$$
$$k_2 = f(x_n + \frac{h}{2}, y_n + h\frac{k_1}{2}),$$
$$k_3 = f(x_n + \frac{h}{2}, y_n + h\frac{k_2}{2}),$$
$$k_4 = f(x_n + h, y_n + hk_3).$$

Further, the approximation technique of the RKCeM(4,4) method can be put in the following form:

$$\tilde{y}_{n+1} = \tilde{y}_n + \frac{2h}{9}\left[\frac{\tilde{k}_1^2+\tilde{k}_1\tilde{k}_2+\tilde{k}_2^2}{\tilde{k}_1+\tilde{k}_2} + \frac{\tilde{k}_2^2+\tilde{k}_2\tilde{k}_3+\tilde{k}_3^2}{\tilde{k}_2+\tilde{k}_3} + \frac{\tilde{k}_3^2+\tilde{k}_3\tilde{k}_4+\tilde{k}_4^2}{\tilde{k}_3+\tilde{k}_4}\right],$$

where

$$\tilde{k}_1 = f(x_n, \tilde{y}_n),$$
$$\tilde{k}_2 = f(x_n + \frac{h}{2}, \tilde{y}_n + h\frac{\tilde{k}_1}{2}),$$
$$\tilde{k}_3 = f(x_n + \frac{h}{2}, \tilde{y}_n + \frac{1}{24}h\tilde{k}_1 + \frac{11}{24}h\tilde{k}_2),$$
$$\tilde{k}_4 = f(x_n + h, \tilde{y}_n + \frac{1}{12}h\tilde{k}_1 - \frac{25}{132}h\tilde{k}_2 + \frac{73}{66}h\tilde{k}_3),$$

The combination of the RKAM(4,4) and RKCeM(4,4) methods is referred to as RKACeM(4,4) and can be formulated as

$$y_{n+1} = y_n + \frac{h}{3}\left[\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2} + \frac{k_3+k_4}{2}\right],$$
$$\tilde{y}_{n+1} = \tilde{y}_n + \frac{2h}{9}\left[\frac{\tilde{k}_1^2+\tilde{k}_1\tilde{k}_2+\tilde{k}_2^2}{\tilde{k}_1+\tilde{k}_2} + \frac{\tilde{k}_2^2+\tilde{k}_2\tilde{k}_3+\tilde{k}_3^2}{\tilde{k}_2+\tilde{k}_3} + \frac{\tilde{k}_3^2+\tilde{k}_3\tilde{k}_4+\tilde{k}_4^2}{\tilde{k}_3+\tilde{k}_4}\right].$$

Error estimate for the explicit RKACeM(4,4) technique

As in Lotkin (1951), an error estimate (ERREST) for the RK method of order four is given by $|\psi(x_n, y_n: h) \leq \frac{73}{720} ML^4|$, where L and M are positive constants. The local truncation error (LTE) of the RKACeM(4,4) method can be estimated as $LTE = y_{n+1} - \tilde{y}_{n+1}$, which may be used to control step size.

The LTE for the RKAM(4,4) method, LTE_{AM}, is given by

$$y_{n+1} = y_{AM} + LTE_{AM} \tag{18}$$

and that for the RKCeM(4,4) method, LTE_{CeM}, is given by

$$y_{n+1} = y_n + LTE_{CeM}, \tag{19}$$

where y_n^{AM}, and y_n^{CeM} are approximated results at x_n attained by the RKAM(4,4) and RKCeM(4,4) methods, respectively. The difference between y_{n+1}^{AM}, and y_{n+1}^{CeM} at x_{n+1} gives an ERREST as

$$y_{n+1}^{AM} - y_{n+1}^{CeM} = LTE_{AM} - LTE_{CeM}. \tag{20}$$

The LTE for the RKAM (4,4) method can be brought to the following form:

$$LTE_{AM} = \frac{h^5}{2880}(24ff_y^4 + f^4 f_{yyyy} + 2f^3 f_y f_{yyy} - 6f^3 f_{yy}^2 + 36f^2 f_y^2 f_{yy}. \tag{21}$$

On the other hand, LTE_{CeM} can be given by

$$LTE_{CeM} = \frac{h^5}{69120}(-762ff_y^4 + 8f^4 f_{yyyy} + 36f^3 f_y f_{yyy} - 744f^3 f_{yy}^2 + 273f^2 f_y^2 f_{yy}. \tag{22}$$

Thus,

$$|LTE_{AM} - LTE_{CeM}| = \frac{h^5}{69120}(186ff_y^4 + 16f^4 f_{yyyy} + 12f^3 f_y f_{yyy} - 600f^3 f_{yy}^2 + 591f^2 f_y^2 f_{yy}. \tag{23}$$

Substituting f, f_y, f_{yy}, etc. in Eq. (23), we have

$$|LTE_{AM} - LTE_{CeM}| = \frac{281}{13824} 186P^5 Q h^5, \tag{24}$$

where $P > 0$ and $Q>0$ are constants satisfying the following conditions:

$|f(x,y)| < Q$ and $\left|\frac{\partial^{i+j} f(x,y)}{\partial^i x\, \partial^j y}\right| < \frac{P^{i+j}}{Q^{j-1}}$,where $i + j \leq 4$.

If we let $TOL = 10^{-5}$, then by setting up $|LTE_{AM} - LTE_{CeM}| < TOL$, the EC and step selection can be determined by Eq. (24) as

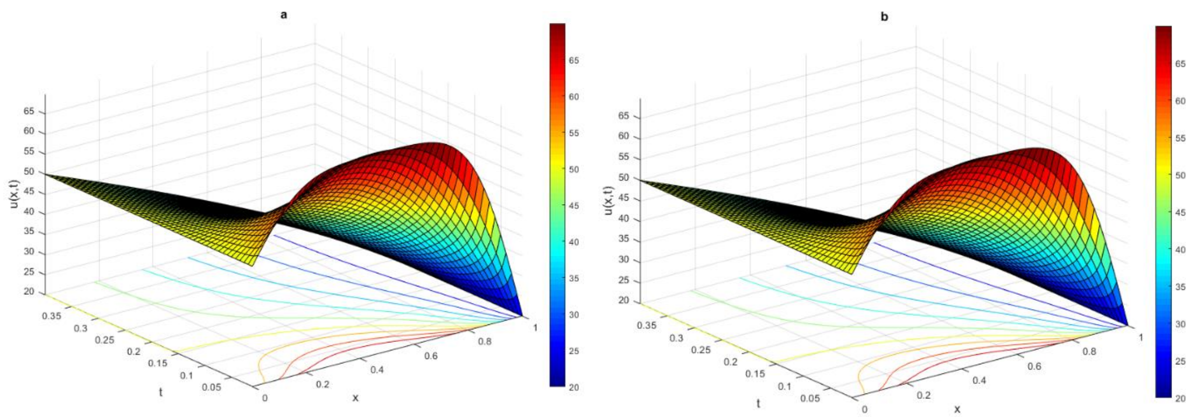
$$\frac{281}{13824}P^5Qh^5 < TOL, \text{ or } h < \left[\frac{49.2\times TOL}{P^5Q}\right]^{\frac{1}{5}}.$$

It is to be notable here that in RKACeM(4,4) method with EC program, we choose the ERREST as the difference between the results attained by the RKAM(4,4) and RKCeM(4,4) methods. From Eq. (24), the ERREST is given by Murugesan et al. (2002)

$$ERREST = |Y_{AM} - Y_{CeM}| \times \frac{281}{13824}. \tag{25}$$

RESULTS AND DISCUSSION

In our study, the numerical calculations are carried out by our developed Matlab routine and the obtained results are compared with those obtained by the GWRM in addition to the RK (4,4) method, the MOLs in coordination with the RK(4,4) method, the FDM and analytic solution. All the mentioned methods are performed for solving the problem taking step size $\Delta x = 0.11$. The temperature profiles at different times obtained through the present study and by employing the other mentioned techniques are depicted in Fig. 1. The figure shows the temperature variation in space over time through 3-dimensional (3D) graphical representation. Our numerical outputs attained by using the methods at $t = 0.01, 0.02, 0.03, 0.05, 0.05$ with analytical solution is presented in Fig. 2 and Table 1 for a better perspective. It is seen from Fig. 2 and Table 1 that the results came out through the approach adopted in this study agree fairly well with those obtained by the other methods and analytical results. However, it is perceived from Fig. 2 and Table 1 that the results obtained via the present study match better with exact solution over the others except nearer to the beginning time. The reason behind the fact may be that in the case of the RKACeM(4,4) method, the adaptive step size (ASS) is advanced in each step of the solution procedure, if necessary, depending upon the accuracy of the results. In advancing step size, a repeated computation may be needed until the wanted result is obtained there. But in the cases of the other mentioned methods, the same step size is maintained throughout a given domain of integration. It is also mentionable here that the same number of time steps has been maintained for both the RKACeM(4,4) and RK(4,4) time integrators.



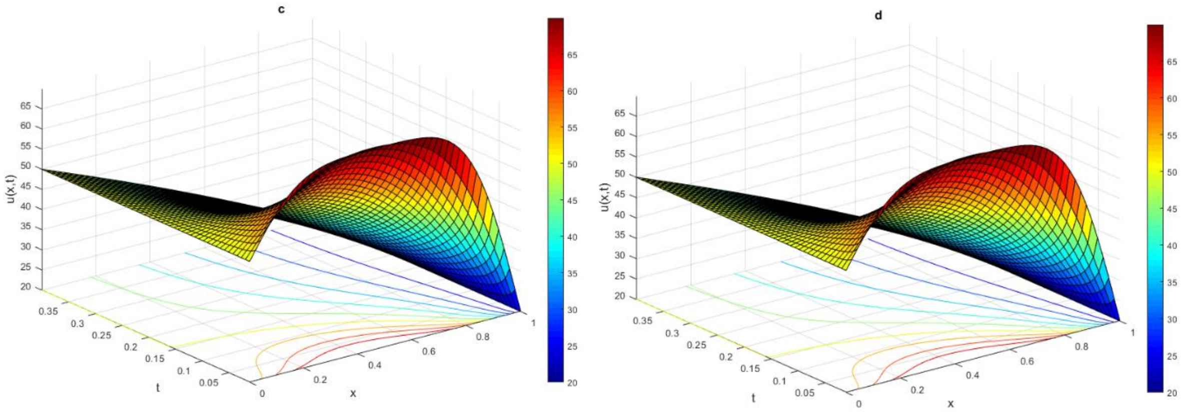


Figure 1: The computed 3D temperature profiles attained via mentioned methods at different points of the domain for chosen times; (a) for the GWRM in addition with the RKACeM(4,4) method; (b) for the GWRM in addition with the RK(4,4) method; (c) for the MOLs in addition with the RK(4,4) method; (d) for FDM.

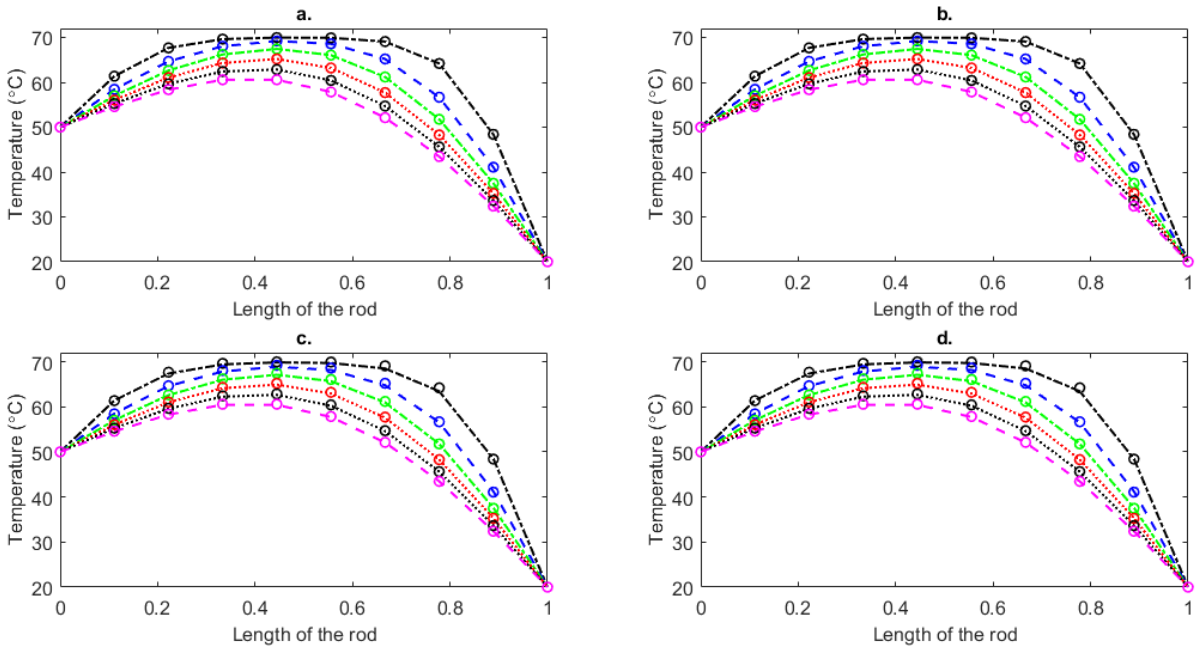


Figure 2: The comparison of attained temperature profiles via mentioned methods at different points of the domain for chosen times with the exact results; (a) for the GWRM in addition with the RKACeM(4,4) method; (b) for the GWRM in addition with the RK(4,4) method; (c) for the MOLs in addition with the RK(4,4) method; (d) for FDM. In each case, the black-dotted-dash curve represents the configuration at time $t = 0.01$, the blue-dash curve at time $t = 0.02$, the greed-dotted-dash curve at time $t = 0.03$, dotted-red curve at time $t = 0.04$, the dotted-black curve at time $t = 0.05$, the dash-magenta curve at time $t = 0.06$, and all small-circles represent the corresponding exact values at the mentioned times.

Figure 3 exhibits the residuals of the results attained in the present study and in the other mentioned approaches with analytic solution at different times. It can be observed from Fig. 3 that the residuals of the computed results using the GWRM (in addition with both RKACeM(4,4) and RK(4,4)) methods from the analytic result are smaller over the MOLs in addition to the RK(4,4) and the FDM which implies the GWRM yields better results over the MOLs and the FDM. To compare the results yielded by the GWRM in addition to both the RKACeM(4,4) and RK(4,4) methods, the residuals of the results attained by the methods are presented in Fig. 4. It can be perceived from Fig. 4 that at the very beginning of the starting time, the GWRM in addition to the RK(4,4) method yields a better result over the present approach (see Fig. 4a). But with a little bit increase in time, residuals can be found to be reduced in the case of the present approach over the GWRM with RK(4,4) time integrator (see Fig. 4). After a short period of time, the variation in residual curve calculated for the present study is about disappeared (see Fig. 4f). It is clear from Figs. 3-4 that the GWRM in addition to the RKACeM(4,4) technique has a smaller variation in error curves over the others. Thus, the approach yields better accuracy in the results after a very short period of time from the starting time.

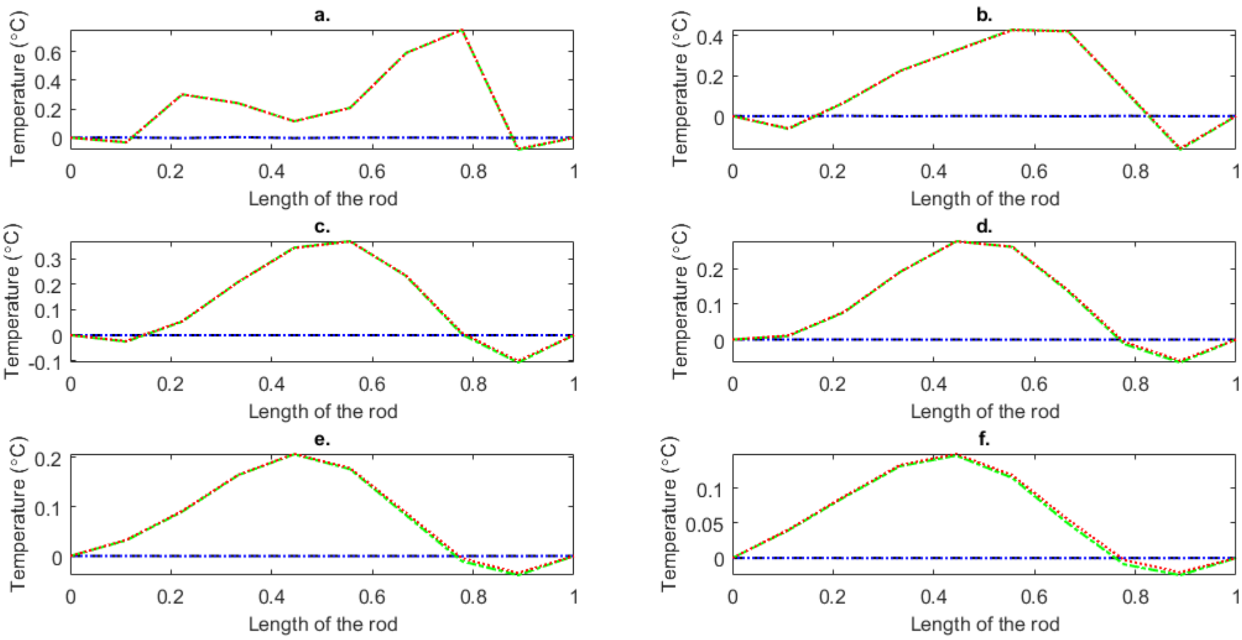


Figure 3: Residuals of the results attained via the mentioned methods at different points of the domain at chosen times; (a) at $t=0.01$; (b) at $t=0.02$; (c) at $t=0.03$; (d) at $t=0.04$; (e) at $t=0.05$; (f) at $t=0.06$. In each case, the black-dash curve represents the residual obtained by the GWRM in addition with the RKACeM(4,4) method, a dotted-blue curve for the GWRM in addition with the RK(4,4) method, a green-dotted-dash curve for the MOLs in addition with the RK(4,4) method, and a dotted-red curve for the FDM.

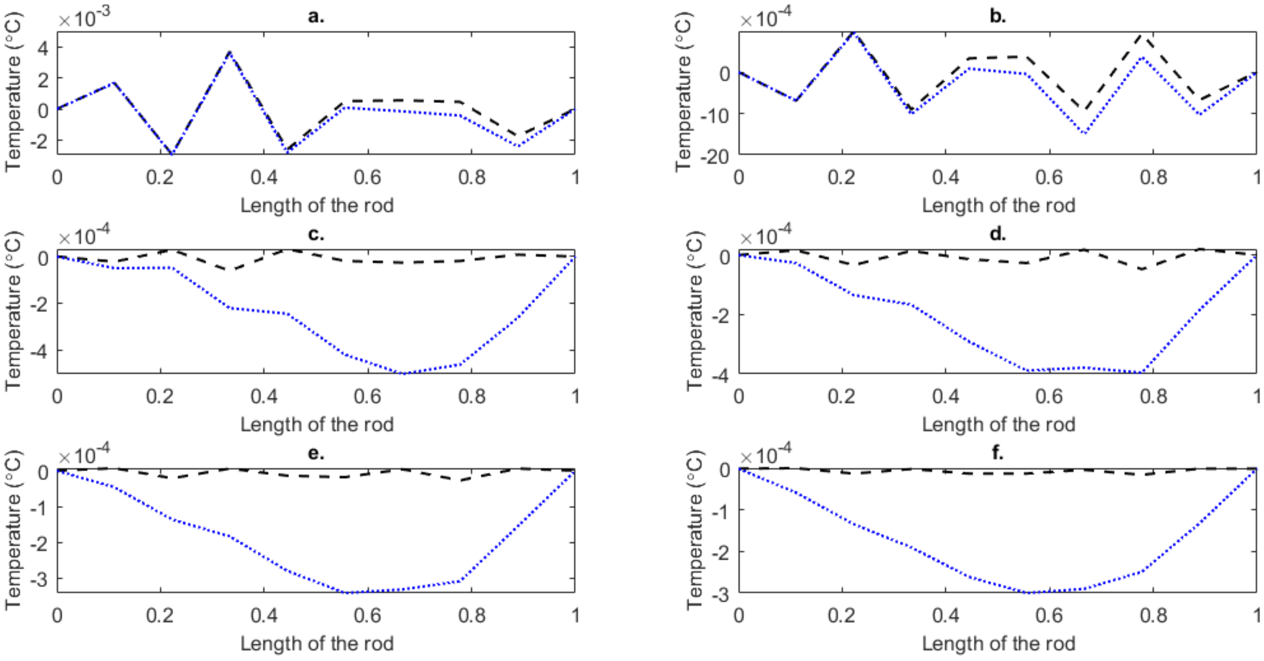


Figure 4: Residual of the results attained by the GWRM in coordination with the RKACeM(4,4) and RK(4,4) time integrators at different points of the domain for chosen times; (a) at $t = 0.01$; (b) at $t = 0.02$; (c) at $t = 0.03$; (d) at $t = 0.04$; (e) at $t = 0.05$; (f) at $t = 0.06$. In each case, the black-dash curve represents the residual obtained for the GWRM in coordination with the RKACeM(4,4) method, a dotted-blue curve for the GWRM in addition with the RK(4,4) method.

Table 1: Comparison of our computed result came out through the present study with the GWRM in addition with the RK (4,4) method, the MOLs with the RK(4,4) method, the FDM and analytic solution at varying time.

t ↓	x→	0	0.1111	0.2222	0.3333	0.4444	0.5556	0.6667	0.7778	0.8889	1	**RMSE values**
	The present study	**50**	**61.3609**	**67.6791**	**69.624**	**69.9746**	**69.9045**	**69.0825**	**64.198**	**48.3973**	**20**	**0.0058**
0.01	GWRM with RK(4,4)	50	61.2327	67.94	69.6745	69.6006	69.9079	69.5130	64.0496	48.1476	20	0.2212
	FDM	50	61.3900	67.3773	69.392	69.8485	69.7083	68.4911	63.4444	48.4751	20	0.3345
	MOL	50	61.3922	67.3776	69.3915	69.848	69.7076	68.49	63.4452	48.4807	20	0.3347
	EXACT	50	61.3588	67.678	69.6314	69.9622	69.9146	69.0788	64.1949	48.3971	20	
	The present study	**50**	**58.4313**	**64.6653**	**68.0482**	**69.2006**	**68.5792**	**65.2072**	**56.6753**	**41.0782**	**20**	**0.0024**
0.02	GWRM with RK(4,4)	50	58.5168	64.6988	67.9499	69.1672	68.6495	65.2307	56.6368	41.0671	20	0.0513
	FDM	50	58.4904	64.5964	67.8215	68.8755	68.1505	64.783	56.5392	41.2347	20	0.2381
	MOL	50	58.4929	64.598	67.821	68.8738	68.1487	64.7831	56.5435	41.2411	20	0.2389
	EXACT	50	58.4293	64.6646	68.0455	69.201	68.5771	65.2038	56.672	41.0741	20	
	The present study	**50**	**56.9842**	**62.6400**	**66.2047**	**67.4418**	**66.0542**	**61.1928**	**51.7568**	**37.4899**	**20**	**0.0008**
0.03	GWRM with	50	57.0199	62.6247	66.1714	67.4633	66.0757	61.1642	51.7544	37.5128	20	0.0221

	RK(4,4)											
	FDM	50	57.0073	62.5845	65.9953	67.0993	65.685	60.9573	51.7466	37.5903	20	0.1910
	MOL	50	57.0093	62.5863	65.9951	67.0976	65.6839	60.9594	51.7524	37.596	20	0.1916
	EXACT	50	56.9839	62.6394	66.2039	67.4415	66.0529	61.1917	51.7555	37.4889	20	
0.04	**The present study**	**50**	**56.032**	**61.062**	**64.3078**	**65.2033**	**63.2058**	**57.703**	**48.2792**	**35.2468**	**20**	**0.0007**
	GWRM with RK(4,4)	50	56.0387	61.0458	64.3033	65.223	63.207	57.6786	48.2839	35.2644	20	0.0128
	FDM	50	56.0204	60.9825	64.1151	64.9247	62.9418	57.5622	48.2846	35.3060	20	0.1459
	MOL	50	56.0219	60.984	64.1152	64.924	62.9423	57.5659	48.2908	35.311	20	0.1457
	EXACT	50	56.0318	61.0615	64.3072	65.2026	63.2047	57.702	48.2781	35.2462	20	
0.05	**The present study**	**50**	**55.269**	**59.6660**	**62.4143**	**62.8559**	**60.4278**	**54.7091**	**45.6136**	**33.6447**	**20**	**0.0006**
	GWRM with RK(4,4)	50	55.2671	59.6546	62.4176	62.8687	60.4237	54.6931	45.6185	33.6553	20	0.0085
	FDM	50	55.236	59.5741	62.2489	62.6479	60.2484	54.6206	45.6172	33.6787	20	0.1097
	MOL	50	55.237	59.5753	62.2496	62.6487	60.2507	54.6254	45.6234	33.6831	20	0.1087
	EXACT	50	55.2688	59.6655	62.4137	62.8551	60.4269	54.7082	45.6127	33.6443	20	

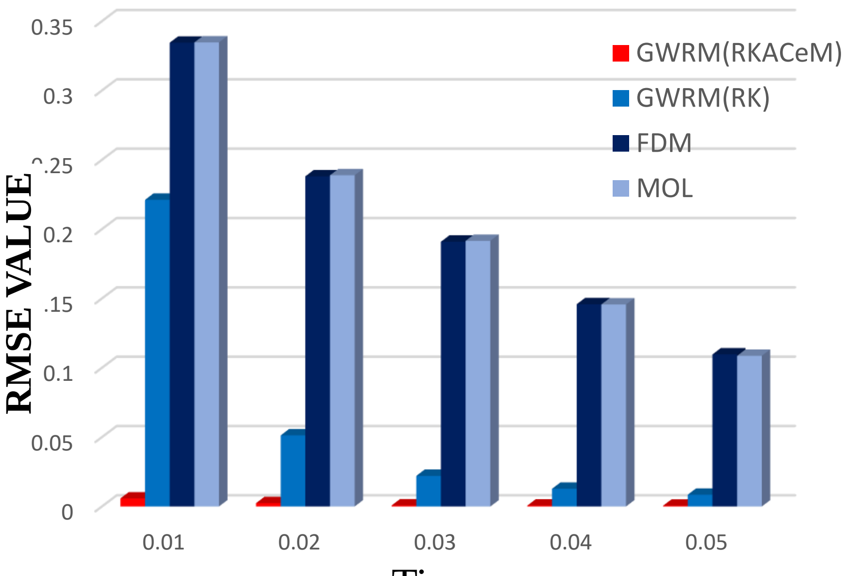


Figure 5: Our estimated RMSE values obtained by using different methods with varying time. The RMSE values are calculated between the model results and analytic solution by the present method, the GWRM in addition with the RK (4,4) method, the MOLs with the RK(4,4) method, and the FDM.

For testing the efficiency of the method employed in the present study, the root mean square errors (RMSEs) have been estimated between computed and analytical results. RMSE values for the results provided by the other techniques have also been evaluated. The results in this regard are presented in Fig. 5. Fig. 5 as well shows the better accuracy of the temporal variations of the results by the present study with exact solutions over the other mentioned methods. Here, it is justified to note that during solving a system of ODEs, the outputs of the system are interdependent. Thus, an error in an output, if available, it can influence the outputs for the next subsequent steps and errors can be piled up. In the embedded RKACeM(4,4) technique, an output with a suitable accuracy can be attained depending upon the tolerance setting. But the other mentioned methods are devoid of providing such a type of advantage. Thus, in the case of accuracy, the approach adopted in the study can defeat the mentioned methods. We also employed the techniques in solving ODHE subject to homogeneous zero BCs and variable type BCs and found that in both the cases the computed results by the approach adopted in this study compared the corresponding exact results better over the other methods mentioned above. The results have not displayed in this communication due to space consumption and we think they are not physically instructive. For testing computational cost, all the codes were run on the same computer (Intel(R) Core(TM) i5-4570, 4^{th} generation) with ASS size 0.11. The CPU time in the case of the employed method used in the study was found to be a little bit more in comparison with that for the other mentioned methods. The reason behind the fact may be that in the case of the embedded system like RKACeM(4,4) method, the ASS can be advanced in each step of the solution procedure depending on the accuracy of results and to attain it, a repeated calculation may be needed. But the computational cost highly depends on the selection of step size. Thus, it can be reduced with setting up of a suitable tolerance. But in the case of the other mentioned methods, such a kind of facility is not available as aforementioned.

CONCLUSION

In the present study, we have solved ODHE with non-homogeneous nonzero Dirichlet's BCs through the GWRM in coordination with the RKACeM(4,4) method and the results are found to be satisfactory with the basis of the RMSE values attained in this study and can be found to be better over the other sophisticated employed methods. Thus, the method is found to produce efficient results with a reasonable cost, which are always stable. The accuracy of the model results can be increased with setting up proper tolerance. Thus, it can be concluded that the different approach adopted in the study is optimal enough to solve ODHE that can be employed to solve such an equation with various types of BCs, which, in turn, can be applied for solving higher dimensional heat equations efficiently.

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