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Bayesian Estimation of Two-Parameters Rayleigh-Logarithmic Using Lindley Approximation

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ABSTRACT

Rayleigh-Logarithmic distribution is used in survival analysis. The main objective of this study is to determine the best estimator for the parameters of this distribution. Estimation methods proposed are Lindley's method under Bayesian framework with two different loss functions; squared error loss function (SELF) and linear exponential loss (LINEX) function and maximum likelihood estimation (MLE). Through a simulation study, the performance of the proposed estimators is compared with respect to their corresponding root mean square error (RMSE). Estimator under SELF is found to be performing better than the estimator under LINEX loss function and the MLE estimators. In conclusion, the estimated parameter under squared error loss function (SELF) is comparatively the best compared to linear exponential (LINEX) loss function and maximum likelihood estimation (MLE).

Keywords: Bayesian estimation, linear exponential (LINEX), Lindley approximation, squared error loss function.

INTRODUCTION

Rayleigh-Logarithmic distribution was introduced by Bugatekin (2017). The paper also discussed the statistical properties of the distribution and proposed a maximum likelihood estimator. This two-parameter mix distribution is useful in survival analysis and reliability theory. The two parameters of Rayleigh-Logarithmic distribution are the shape parameter (p) and the scale parameter (σ^2). Hameed and Alwan (2020) had conducted a study of the reliability and hazard function of this distribution.

This study proposed Bayesian estimation of the parameters. Estimating two-parameters under Bayesian estimation framework would requires multiple integrations and is complicated to solve. Hence, Lindley approximation is used to solve such cases. Soliman et al. (2010) compared maximum likelihood and Bayesian estimators of the inverse Rayleigh distribution. Similar work can be found in Restogi and Merovci (2018) where Lindley's approximation was used as the estimator for parameters a three-parameters Weibull Rayleigh distribution.

In this study, estimation methods considered are the maximum likelihood method and Bayesian estimation with square error loss function and linear exponential loss function (Zellner, 1986).

The rest of the paper is organized as follows: The model and the maximum likelihood estimation is discussed in the next section, followed by the Lindley approximation method in the following section. A simulation study is performed using proposed methods and results of their corresponding root mean square error is used to investigate their performances. The results and discussions are given in the remaining sections.

METHODOLOGIES

Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) is a technique that defines values for the parameters of a model of a given distribution. The probability density function and cumulative distribution function for the two-parameters Rayleigh-Logarithmic distribution are:

$$f(x,\theta) = -\frac{x}{\sigma^2 \ln(1-p)} e^{\frac{-x^2}{2\sigma^2}} p \left(1 - e^{\frac{-x^2}{2\sigma^2}} p\right)^{-1}, \sigma^2 > 0, 0 < p < 1 \tag{1}$$

$$F(x,\theta) = 1 - \frac{\ln\left(1 - e^{\frac{-x^2}{2\sigma^2}} p\right)}{\ln(1-p)} \tag{2}$$

where $\theta = (\sigma^2, p)$.

The likelihood function of Rayleigh-Logarithmic is:

$$L(x,\theta) = \prod_{i=1}^{n} \left(-\frac{x}{\sigma^2 \ln(1-p)} e^{\frac{-x^2}{2\sigma^2}} p \left(1 - e^{\frac{-x^2}{2\sigma^2}} p\right)^{-1}\right) \tag{3}$$

Taking logarithm of Equation 3, we obtained,

$$\ell(x,\theta) = \sum_{t=1}^{n} \ln(x_t) + \ln(-1)^n - n\ln(\sigma^2) - n\ln(\ln(1-p)) - \sum_{t=1}^{n} \frac{x_t^2}{2\sigma^2} + n\ln(p) - \sum_{t=1}^{n} \ln\left(1 - e^{\frac{x_t^2}{2\sigma^2}} p\right) \tag{4}$$

Maximum likelihood estimators for parameters σ^2 and p are obtained by maximizing Equation 4 with respect to σ^2 and p respectively, such that;

$$-\frac{4n}{\sigma}+\sum_{t=1}^{n}\frac{x_t^2}{2\sigma^4}-\sum_{t=1}^{n}\left(\frac{\frac{px_t^2}{2\sigma^4}e^{\frac{-x_t^2}{2\sigma^2}}}{1-e^{\frac{-x_t^2}{2\sigma^2}}p}\right)=0 \tag{5}$$

and

$$\frac{n}{(1-p)\ln(1-p)}+\frac{n}{p}+\sum_{t=1}^{n}\frac{e^{\frac{-x_t^2}{2\sigma^2}}}{1-e^{\frac{-x_t^2}{2\sigma^2}}p}=0 \tag{6}$$

Therefore, $\widehat{\sigma^2}$ and $\hat{p}$ can be obtained by solving Equations 9 and 10 iteratively.

Bayesian Estimation

Defining a prior distribution is crucial in Bayesian framework. Guure et al. (2012) state that in Bayesian analysis, non-informative prior can be used if the prior knowledge about the parameter is not available. Since there is no knowledge of the parameters, the extension of Jeffreys' prior is used. Jeffrey's prior is derived using the Fisher information matrix;

$$u(\sigma^2,p)\propto\left(\frac{1}{\sigma^2 p}\right)$$

Since the Bayes estimators cannot be obtained in closed form, Lindley's approximation is used to calculate the Bayesian posterior estimates. Besides prior, defining loss function is also important. Loss function is a function of a difference between estimated and true values of data. According to Dey (2012), estimation and prediction of loss function might be a problem because there is no specific analytical procedure to identify the appropriate loss function. In this study, we take into account symmetric and asymmetric loss functions i.e., squared error loss function (SELF) and linear exponential loss function (LINEX) (Varian, 1975) respectively.

The squared error loss function is given by $L(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2$ where $\hat{\theta}$ is the estimator of θ. The LINEX loss function is defined as $L(\Delta)=e^{a\Delta}-a\Delta-1,\ a\neq 0$ where $\Delta=\hat{\theta}-\theta$ and $\hat{\theta}$ is the estimator of θ and a is the slope of the loss function.

According to Guure and Ibrahim (2014), Bayes estimator with SELF, $\hat{u}_{BS}$ is the posterior mean. $\hat{u}_{BS}$ is a function $u=(\sigma^2,p)$ of the unknown parameters σ^2 and p. Therefore, with Lindley approximation the SELF-Bayes estimator is defined as

$$\widehat{\theta_{BS}}=u+\frac{1}{2}[(u_{11}\sigma_{11}+u_{22}\sigma_{22})]+u_1p_1\sigma_{11}+u_2p_2\sigma_{22}$$

$$+\frac{1}{2}[(L_{30}u_1\sigma_{11}^2)+(L_{03}u_2\sigma_{22}^2)]$$

where,

$$p_1 = -\frac{1}{\sigma^2}\ , p_2 = -\frac{1}{p}$$

$$\sigma_{11} = -L_{20}^{-1}\ , \sigma_{22} = -L_{02}^{-1}$$

When $u = \sigma^2$,

$$u_1 = \frac{\delta u}{\delta \sigma^2} = 1, u_{11} = \frac{\delta^2 u}{(\delta \sigma^2)^2} = 0$$

$$u_2 = \frac{\delta u}{\delta p} = 0, u_{22} = \frac{\delta^2 u}{(\delta p)^2} = 0$$

When $u = p$,

$$u_1 = \frac{\delta u}{\delta \sigma^2} = 0, u_{11} = \frac{\delta^2 u}{(\delta \sigma^2)^2} = 0$$

$$u_2 = \frac{\delta u}{\delta p} = 1, u_{22} = \frac{\delta^2 u}{(\delta p)^2} = 0$$

Posterior expectation of the LINEX loss function is given by

$$E_\theta L(\Delta) = b\left[e^{a\hat{\theta}} E_\theta e^{-a\theta} - a\left(\hat{\theta} - E_\theta \theta\right) - 1\right]. \quad (7)$$

By minimizing Equation 7, the value of the posterior mean is:

$$\widehat{\theta_{BL}} = -\frac{1}{a} \ln E_\theta (e^{-a\theta})$$

The LINEX loss function is obtained by using the same Lindley procedure.

$$\widehat{u_{BL}} = u + \frac{1}{2}[(u_{11}\sigma_{11} + u_{22}\sigma_{22})] + u_1 p_1 \sigma_{11} + u_2 p_2 \sigma_{22}$$

$$+ \frac{1}{2}[(L_{30} u_1 \sigma_{11}^2) + (L_{03} u_2 \sigma_{22}^2)]$$

When $u = e^{-a\sigma^2}$,

$$u_1 = \frac{\delta u}{\delta \sigma^2} = -a e^{-a\sigma^2}, u_{11} = \frac{\delta^2 u}{(\delta \sigma^2)^2} = a^2 e^{-a\sigma^2}$$

$$u_2 = \frac{\delta u}{\delta p} = 0, u_{22} = \frac{\delta^2 u}{(\delta p)^2} = 0$$

When $u = e^{-ap}$,

$$u_1 = \frac{\delta u}{\delta \sigma^2} = 0, u_{11} = \frac{\delta^2 u}{(\delta \sigma^2)^2} = 0$$

$$u_2 = \frac{\delta u}{\delta p} = -ae^{-ap}, u_{22} = \frac{\delta^2 u}{(\delta p)^2} = a^2 e^{-ap}$$

Simulation Study

To compare the maximum likelihood estimators of the parameters of the Rayleigh-Logarithmic distribution, $\widehat{\sigma^2}_{ML}\ and\ \hat{p}_{ML}$ with the Bayesian estimators, $\widehat{\sigma^2}_{ML}\ and\ \hat{p}_{ML}$ we run a simulation study. We considered $\sigma^2 = 0.5, 1.0, 1.5$ and $p = 0.5, 0.1, 0.9$. With LINEX loss function, $a = 0.6$ and -0.6 are used. Data of size $n = 25, 50,$ and100 is generated from the Rayleigh-Logarithmic distribution using the inverse transform technique. The sample size is chosen to represent small, medium and large data sets. MLE and Lindley approximation were performed on these data sets for 5000 iterations. Comparison between the estimators is made using root mean square error (RMSE). Conclusions are given regarding the behavior of the estimators.

RESULTS AND DISCUSSION

Table 1 summaries the values of root mean square error (RMSE) calculated for the Bayesian estimators with squared error loss function and LINEX loss function and the maximum likelihood estimator.

In every case, the Bayes estimator under SELF of the scale parameter, $\widehat{\sigma^2}$ produces the lowest RMSE in comparison to when using LINEX loss function and maximum likelihood estimation indicating that this estimator performs better than others. Note that with LINEX loss function, the RMSE values are similar for $a = 0.6$ and $a = -0.6$. This result supports our finding that an asymmetrical loss function is a better choice.

For the shape parameter, p the smallest RMSE are obtained from Bayesian estimation under SELF. This suggest that this estimator provides better estimates compared to Bayes estimator with LINEX loss function and the maximum likelihood estimator. It is also found that the RMSE values decrease as the sample sizes increase.

Table 1. Root mean square error (RMSE) of parameter estimates $\widehat{\sigma^2}$ and $\hat{p}$, with respect to Bayes estimator with SELF, LINEX loss function and maximum likelihood estimator (MLE) at $n = 25, 50 \text{ and } 100$.

			SELF		LINEX $[a = 0.6]$		LINEX $[a = -0.6]$		MLE	
n	σ^2	p	$\widehat{\sigma^2}$	$\hat{p}$	$\widehat{\sigma^2}$	$\hat{p}$	$\widehat{\sigma^2}$	$\hat{p}$	$\widehat{\sigma^2}$	$\hat{p}$
25	0.5	0.1	0.0042	0.0004	0.0061	0.0005	0.0060	0.0005	0.0261	0.4129
		0.5	0.0047	0.0183	0.0062	0.0204	0.0061	0.0206	0.0233	0.2297
		0.9	0.0038	0.0161	0.0050	0.0165	0.0050	0.0167	0.0209	0.0582
	1.0	0.1	0.0060	0.0004	0.0115	0.0005	0.0114	0.0005	0.0379	0.3006
		0.5	0.0064	0.0183	0.0114	0.0206	0.0113	0.0206	0.0338	0.1665
		0.9	0.0053	0.0161	0.0093	0.0167	0.0093	0.0167	0.0300	0.0418
	1.5	0.1	0.0080	0.0004	0.0182	0.0005	0.0180	0.0005	0.0470	0.2486
		0.5	0.0144	0.0183	0.0225	0.0204	0.0218	0.0206	0.0417	0.1374
		0.9	0.0166	0.0161	0.0228	0.0165	0.0216	0.0167	0.0369	0.0344
50	0.5	0.1	0.0020	0.0002	0.0029	0.0002	0.0029	0.0002	0.0125	0.1900
		0.5	0.0022	0.0091	0.0030	0.0102	0.0030	0.0103	0.0112	0.1054
		0.9	0.0018	0.0079	0.0025	0.0081	0.0025	0.0081	0.0101	0.0263
	1.0	0.1	0.0028	0.0002	0.0056	0.0002	0.0056	0.0002	0.0182	0.1386
		0.5	0.0029	0.0091	0.0056	0.0102	0.0056	0.0103	0.0162	0.0764
		0.9	0.0026	0.0079	0.0048	0.0081	0.0047	0.0081	0.0144	0.0189
	1.5	0.1	0.0035	0.0002	0.0088	0.0002	0.0087	0.0002	0.0226	0.1147
		0.5	0.0047	0.0091	0.0094	0.0102	0.0093	0.0103	0.0201	0.0631
		0.9	0.0052	0.0079	0.0089	0.0081	0.0088	0.0081	0.0177	0.0155
100	0.5	0.1	0.0010	0.0001	0.0014	0.0001	0.0014	0.0001	0.0099	0.1475
		0.5	0.0010	0.0046	0.0015	0.0051	0.0015	0.0051	0.0088	0.0812
		0.9	0.0009	0.0039	0.0013	0.0040	0.0013	0.0040	0.0079	0.0201
	1.0	0.1	0.0013	0.0001	0.0028	0.0001	0.0028	0.0001	0.0141	0.1056
		0.5	0.0014	0.0046	0.0028	0.0051	0.0028	0.0051	0.0126	0.0580
		0.9	0.0013	0.0039	0.0025	0.0040	0.0025	0.0040	0.0112	0.0143
	1.5	0.1	0.0017	0.0001	0.0043	0.0001	0.0043	0.0001	0.0174	0.0867
		0.5	0.0019	0.0046	0.0044	0.0051	0.0044	0.0051	0.0155	0.0476
		0.9	0.0020	0.0039	0.0042	0.0040	0.0041	0.0040	0.0137	0.0117

CONCLUSION

As a conclusion, for the shape and scale parameters of Rayleigh Logarithmic distribution, the Bayes estimators under squared error loss function is the best compared to when using LINEX loss function and the maximum likelihood estimator. The estimators also perform better with larger sample size, n.

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