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Growth of Solutions for a Coupled Viscoelastic Kirchhoff System with Distributed Delay Terms

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ABSTRACT

In this work, we are concerned with a coupled nonlinear viscoelastic Kirchhoff system with distributed delay terms and source terms. Under suitable conditions, we prove the exponential growth of solutions.

Keywords: Exponential growth, Distributed delay, Nonlinear source.

INTRODUCTION

In this paper, we investigate the following viscoelastic Kirchhoff system with distributed delay terms and source terms:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t(x,t) \\ +\int_{\tau_1}^{\tau_2} |\mu_2(\varsigma)| u_t(x,t-\varsigma)\,d\varsigma = f_1(u,v),\ (x,t)\in\Omega\times R_+, \\ v_{tt} - M\left(\|\nabla v\|^2\right)\Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + \mu_3 v_t(x,t) \\ +\int_{\tau_1}^{\tau_2} |\mu_4(\varsigma)| v_t(x,t-\varsigma)\,d\varsigma = f_2(u,v),\ (x,t)\in\Omega\times R_+, \\ u(x,t)=0,\ \mathrm{v}(x,t)=0,\ \mathrm{x}\in\partial\Omega, \\ u_t(x,-t) = f_0(x,t),\ \mathrm{v}_t(x,-t)=k_0(x,t),\ (x,t)\in\Omega\times(0,\tau_2), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \mathrm{x}\in\Omega, \\ v(x,0)=v_0(x),\ \mathrm{v}_t(x,0)=v_1(x),\ \mathrm{x}\in\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n, with a smooth boundary $\partial\Omega$. μ_1, $\mu_3>0$, τ_1, τ_2 are the time delay with $0\le\tau_1<\tau_2$, μ_2, μ_4 are L^∞ functions, and g, h are differential functions. $M(s)$ is a C^1 nonnegative function on R^+ defined as $M(s)=m_0+\alpha s^\gamma$ with $m_0>0$, $\alpha\ge 0$ and $\gamma\ge 0$, to simplify our calculations we take $M(s)=1+s^\gamma$ in the problem (1).

From mathematically point of view, ''Growth'' phenomenon gives us importance knowledge to understand the asymptotic behaviour of the equation when time arrives at infinity. In recent years, there has been published much work concerning the wave equation with time delay or time varying delay. Our aim is to study the exponential growth of solutions for viscoelastic Kirchhoff system with distributed delay terms.

The viscous materials are the converse of elastic materials that possess the ability to dissipate and store mechanical energy. Because of the mechanical properties of these viscous substances are of great significance when they seem in many applications of natural sciences [14].
Time delays often appear in many practical problems such as thermal, economic phenomena, biological, chemical and physical. The authors, in [2], indicated that a small delay in a boundary control could turn a well-behave hyperbolic system into a wild one, hence, delay becomes a source of instability. Besides, sometimes it can also be improved the performance of the system.
Rahmoune et al. [14], considered the following Klein-Gordon system with strong damping, nonlinear source and distributed delay terms:

$$\begin{cases} u_{tt}+m_1u^2-\Delta u-\omega_1\Delta u_t+\int_0^t g(t-s)\Delta u(s)\,ds+\mu_1 u_t \\ +\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|u_t(x,t-\varsigma)\,d\varsigma = f_1(u,v),\ (x,t)\in\Omega\times R_+, \\ v_{tt}+m_2v^2-\Delta v-\omega_2\Delta v_t+\int_0^t h(t-s)\Delta v(s)\,ds+\mu_3 v_t \\ +\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|v_t(x,t-\varsigma)\,d\varsigma = f_2(u,v),\ (x,t)\in\Omega\times R_+, \end{cases} \tag{2}$$

where $m_1,\ \mathrm{m}_2,\ \omega_1,\ \omega_2>0$. The authors investigated the exponential growth of solutions for the problem (2) under suitable conditions.
The viscoelastic wave equation of the form:

$$u_{tt}-\Delta u+\int_0^t g(t-\tau)\Delta u(\tau)\,d\tau+h(u_t)=f(u),\ \mathrm{x}\in\Omega,\ \mathrm{t}>0, \tag{3}$$

has been investigated extensively by some authors (see [3], [6], [7], [15], and references therein).
In [4], Mezouar proved the global existence and decay properties of solutions for the following viscoelastic Kirchhoff equation:

$$\begin{aligned} &[u_t]^l u_{tt}-M\left(\|\nabla u\|^2\right)\Delta u-\Delta u_{tt}+\int_0^t h(t-s)\Delta u(s)\,ds \\ &\quad+\alpha u+\mu_1 g_1(u_t(x,t))+\mu_2 g_2\left(u_t(x,t-\tau(t))\right) \\ &=0. \end{aligned} \tag{4}$$

Moreover, in [5], she established the global existence and exponential decay of solutions for generalized coupled Kirchhoff system with a time varying delay term.
In [12], Pişkin considered the following system of viscoelastic wave equations with weak damping terms:

$$\begin{cases} u_{tt}-\Delta u+\int_0^t g_1(t-\tau)\Delta u(\tau)\,d\tau+u_t = f_1(u,v) \\ v_{tt}-\Delta v+\int_0^t g_2(t-\tau)\Delta v(\tau)\,d\tau+v_t = f_2(u,v). \end{cases} \tag{5}$$

He obtained the global nonexistence of solutions for the problem (5). Moreover, in [11], Pişkin proved the blow up of solutions for coupled nonlinear Klein-Gordon equations with weak damping terms. Also, Pişkin, in [13], established the decay estimates of the solution by using Nakao's inequality and proved the blow up of solution in a finite time with negative initial energy.
Motivated by previous works, we prove the exponential growth of solutions of the system (1) both Kirchhoff term ($M\left(\|\nabla u\|^2\right)$) that depends on ($\|\nabla u\|^2$) and viscoelastic term ($\int_0^t g(t-s)\Delta u(s)\,ds$) with distributed delay terms ($\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|u_t(x,t-\varsigma)\,d\varsigma$) in the system, and in a similar way, we benefit from the study of [14]. We investigated that the solutions of system (1) grows exponentially under suitable conditions.

The content of this paper is organized as follows: In section 2, we provide some assumptions and lemmas that will be used later. In section 3, we prove our main result that we will get the growth result for the system (1).

PRELIMINARIES

In this part, we give some assumptions and lemmas which will be used throught this work. Firstly, we make the following assumptions:

(A1) g, h: $\mathrm{R}_+ \to \mathrm{R}_+$ are differential decreasing functions such that

$$\begin{cases} g(t) \geq 0,\ 1\text{-}\int_0^\infty g(s)\,ds = l_1 > 0, \\ h(t) \geq 0,\ 1\text{-}\int_0^\infty g(s)\,ds = l_2 > 0. \end{cases} \tag{6}$$

(A2) There exist constants $\xi_1,\ \xi_2$ such that

$$\begin{cases} g'(t) \leq -\xi_1 g(t),\ \mathrm{t} \geq 0, \\ h'(t) \leq -\xi_2 h(t),\ \mathrm{t} \geq 0. \end{cases} \tag{7}$$

(A3) $\mu_2,\ \mu_4 : [\tau_1, \tau_2] \to R$ are L^∞ functions so that, for all $\delta > \frac{1}{2}$,

$$\begin{cases} \left(\frac{2\delta - 1}{2}\right)\int_{\tau_1}^{\tau_2} |\mu_2(\varsigma)|\,d\varsigma < \mu_1, \\ \left(\frac{2\delta - 1}{2}\right)\int_{\tau_1}^{\tau_2} |\mu_4(\varsigma)|\,d\varsigma < \mu_3. \end{cases} \tag{8}$$

Concerning the functions $f_1(u,v)$ and $f_2(u,v)$, we take the source terms as follows:

$$\begin{cases} f_1(u,v) = a_1 |u+v|^{2(p+1)}(u+v) + b_1 |u|^p u |v|^{p+2}, \\ f_2(u,v) = a_1 |u+v|^{2(p+1)}(u+v) + b_1 |v|^p v |u|^{p+2}, \end{cases} \tag{9}$$

where $a_1,\ \mathrm{b}_1 > 0$.

We have the following lemmas.

Lemma 1 ([14]) There exists a function $F(u,v)$ such that

$$\begin{aligned} F(u,v) &= \frac{1}{2(p+2)}\left[u f_1(u,v) + v f_2(u,v)\right] \\ &= \frac{1}{2(p+2)}\left[a_1 |u+v|^{2(p+2)} + 2b_2 |uv|^{p+2}\right] \\ &\geq 0, \end{aligned} \tag{10}$$

where

$$\frac{\partial F}{\partial u} = f_1(u,v),\ \frac{\partial F}{\partial v} = f_2(u,v),$$

taking $a_1 = b_1 = 1$ for convenience.

Lemma 2 ([8]) There exist two positive constants c_0 and c_1 such that

$$\frac{c_0}{2(p+2)}\left(|u|^{2(p+2)} + |v|^{2(p+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(p+2)}\left(|u|^{2(p+2)} + |v|^{2(p+2)}\right). \tag{11}$$

Lemma 3 ([10]) For $\phi,\ \psi \in \mathrm{C}^1(R_+, R)$ we have

$$\int_\Omega \phi * \psi \psi_t dx = -\frac{1}{2}\phi(t)\|\psi(t)\|^2 + \frac{1}{2}(\phi' \circ \psi)(t) - \frac{1}{2}\frac{d}{dt}\left[(\phi\circ\psi)(t) - \left(\int_0^t \phi(s)ds\right)\|\psi\|^2\right] \quad (12)$$

where

$$(\phi\circ\psi)(t) = \int_0^t \phi(t-s)\|\psi(t)-\psi(s)\|^2 ds.$$

GROWTH RESULT

In this part, we prove the exponential growth of solutions of the problem (1). In order to do so, we first introduce, as in [9], the new variables:

$$y(x,\rho,\varsigma,t) = u_t(x,t-\varsigma\rho),$$
$$z(x,\rho,\varsigma,t) = v_t(x,t-\varsigma\rho),$$

thus we have,

$$\begin{aligned} \varsigma y_t(x,\rho,\varsigma,t) + y_\rho(x,\rho,\varsigma,t) = 0,\\ y(x,0,\varsigma,t) = u_t(x,t), \end{aligned} \quad (13)$$

and

$$\begin{aligned} \varsigma z_t(x,\rho,\varsigma,t) + z_\rho(x,\rho,\varsigma,t) = 0,\\ z(x,0,\varsigma,t) = v_t(x,t). \end{aligned} \quad (14)$$

Hence, problem (1) is equivalent to

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t(x,t) \\ +\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|\, y(x,1,\varsigma,t)d\varsigma = f_1(u,v),\ \mathrm{x}\in\Omega,\ t\ge 0, \\ v_{tt} - M\left(\|\nabla v\|^2\right)\Delta v + \int_0^t h(t-s)\Delta v(s)ds + \mu_3 v_t(x,t) \\ +\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|\, z(x,1,\varsigma,t)d\varsigma = f_2(u,v),\ \mathrm{x}\in\Omega,\ t\ge 0, \\ \varsigma y_t(x,\rho,\varsigma,t) + y_\rho(x,\rho,\varsigma,t) = 0, \\ \varsigma z_t(x,\rho,\varsigma,t) + z_\rho(x,\rho,\varsigma,t) = 0, \end{cases} \quad (15)$$

with the initial and boundary condition

$$\begin{cases} u(x,t) = 0,\ \mathrm{v}(x,t) = 0,\ \mathrm{x}\in\partial\Omega, \\ y(x,\rho,\varsigma,0) = f_0(x,\varsigma\rho),\ \mathrm{z}(x,\rho,\varsigma,0) = k_0(x,\varsigma\rho), \\ u(x,0) = u_0(x),\ \mathrm{u}_t(x,0) = u_1(x), \\ v(x,0) = v_0(x),\ \mathrm{v}_t(x,0) = v_1(x), \end{cases} \quad (16)$$

where

$$(x,\rho,\varsigma,t) \in \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

We define the functional space $\hat{\mathrm{H}}$ as

$$\hat{\mathrm{H}} = H_0^1(\Omega)\times L^2(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega\times(0,1))\times(\tau_1,\tau_2)$$
$$\times L^2(\Omega\times(0,1))\times(\tau_1,\tau_2).$$

Theorem 4 ([14]) Suppose that (6), (7) and (8) hold. Let us

$$\begin{aligned} -1 < p < \frac{4-n}{n-2},\ \mathrm{n}\ge 3, \\ p \ge -1,\ \mathrm{n}=1,2. \end{aligned} \quad (17)$$

Then, for any initial data, $(u_0,\ u_1,\ \mathrm{v}_0,\ \mathrm{v}_1,\ f_0,\ \mathrm{k}_0) \in \mathrm{H},$ the problem (15) has a unique solution, in $C([0,T];\mathrm{H})$ for some $T>0.$

Lemma 5 Suppose that (6), (7), (8) and (17) hold, let (u,v,y,z) be a solution of (15), then $E(t)$ is nonincreasing,

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|v_t\|^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+\frac{1}{2(\gamma+1)}\|\nabla v\|^{2(\gamma+1)}$$
$$+\frac{1}{2}l_1\|\nabla u\|^2+\frac{1}{2}l_2\|\nabla v\|^2+\frac{1}{2}(go\nabla u)+\frac{1}{2}(ho\nabla v)+\frac{1}{2}M(y,z) \qquad (18)$$
$$-\int_\Omega F(u,v)\,dx$$

satisfies

$$E'(t)\le -c_3\left\{\|u_t\|^2+\|v_t\|^2+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|\,y^2(x,1,\varsigma,t)\,d\varsigma dx+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|\,z^2(x,1,\varsigma,t)\,d\varsigma dx\right\} \qquad (19)$$
$$\le 0,$$

where

$$M(y,z)=\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varsigma\left\{|\mu_2(\varsigma)|\,y^2(x,\rho,\varsigma,t)+|\mu_4(\varsigma)|\,z^2(x,\rho,\varsigma,t)\right\}d\varsigma\, d\rho\, dx. \qquad (20)$$

Proof. We multiply the first and the second equation in (15) respectively by u_t, v_t and integrating over Ω, we obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|v_t\|^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+\frac{1}{2(\gamma+1)}\|\nabla v\|^{2(\gamma+1)}\right\}$$
$$\frac{d}{dt}\left\{\frac{1}{2}l_1\|\nabla u\|^2+\frac{1}{2}l_2\|\nabla v\|^2+\frac{1}{2}(go\nabla u)+\frac{1}{2}(ho\nabla v)-\int_\Omega F(u,v)\,dx\right\}$$
$$=-\mu_1\|u_t\|^2-\int_\Omega u_t\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|\,y(x,1,\varsigma,t)\,d\varsigma dx$$
$$-\mu_3\|v_t\|^2-\int_\Omega v_t\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|\,z(x,1,\varsigma,t)\,d\varsigma dx \qquad (21)$$
$$+\frac{1}{2}(g'o\nabla u)-\frac{1}{2}g(t)\|\nabla u\|^2$$
$$+\frac{1}{2}(h'o\nabla v)-\frac{1}{2}h(t)\|\nabla v\|^2,$$

and by using the initial and boundary conditions in (15), we have

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2}\varsigma\left|\mu_2(\varsigma)\right|y^2(x,\rho,\varsigma,t)d\varsigma d\rho dx\\
&=-\frac{1}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2}2\left|\mu_2(\varsigma)\right|yy_{\rho}d\varsigma d\rho dx\\
&=\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(\varsigma)\right|y^2(x,0,\varsigma,t)d\varsigma dx\\
&-\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(\varsigma)\right|y^2(x,1,\varsigma,t)d\varsigma dx\\
&=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu_2(\varsigma)d\varsigma\right)\|u_t\|^2\\
&-\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(\varsigma)\right|y^2(x,1,\varsigma,t)d\varsigma dx
\end{aligned}\tag{22}$$

and

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2}\varsigma\left|\mu_4(\varsigma)\right|z^2(x,\rho,\varsigma,t)d\varsigma d\rho dx\\
&=-\frac{1}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2}2\left|\mu_4(\varsigma)\right|zz_{\rho}d\varsigma d\rho dx\\
&=\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_4(\varsigma)\right|z^2(x,0,\varsigma,t)d\varsigma dx\\
&-\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_4(\varsigma)\right|z^2(x,1,\varsigma,t)d\varsigma dx\\
&=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu_4(\varsigma)d\varsigma\right)\|v_t\|^2\\
&-\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_4(\varsigma)\right|z^2(x,1,\varsigma,t)d\varsigma dx,
\end{aligned}\tag{23}$$

then

$$\begin{aligned}
\frac{d}{dt}E(t)&=-\mu_1\|u_t\|^2-\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(\varsigma)\right|u_t y(x,1,\varsigma,t)d\varsigma dx+\frac{1}{2}(g' o\nabla u)\\
&-\frac{1}{2}g(t)\|\nabla u\|^2+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu_2(\varsigma)d\varsigma\right)\|u_t\|^2\\
&-\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(\varsigma)\right|y^2(x,1,\varsigma,t)d\varsigma dx\\
&-\mu_3\|v_t\|^2-\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_4(\varsigma)\right|v_t z(x,1,\varsigma,t)d\varsigma dx+\frac{1}{2}(h' o\nabla u)\\
&-\frac{1}{2}h(t)\|\nabla v\|^2+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu_4(\varsigma)d\varsigma\right)\|v_t\|^2\\
&-\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_4(\varsigma)\right|z^2(x,1,\varsigma,t)d\varsigma dx.
\end{aligned}\tag{24}$$

From (21)-(23), we obtain (18). Moreover, utilizing Young's inequality, (6), (7) and (8) in (24), we get (19).
Next, we define the functional

$$\begin{aligned} H(t)=-E(t)=&-\frac{1}{2}\|u_t\|^2-\frac{1}{2}\|v_t\|^2-\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} \\ &-\frac{1}{2(\gamma+1)}\|\nabla v\|^{2(\gamma+1)}-\frac{1}{2}l_1\|\nabla u\|^2-\frac{1}{2}l_2\|\nabla v\|^2 \\ &-\frac{1}{2}(go\nabla u)-\frac{1}{2}(ho\nabla v)-\frac{1}{2}M(y,z) \\ &+\frac{1}{2(p+2)}\left[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{2(p+2)}^{2(p+2)}\right]. \end{aligned} \tag{25}$$

Theorem 6 Suppose that (6)-(8) and (17) hold. Suppose further that $E(0)<0$, then the solution of problem (15) grows exponentially.

Proof. By (18), we have

$$E(t)\le E(0)<0. \tag{26}$$

Hence,

$$\begin{aligned} H'(t)&=-E'(t) \\ &\ge c_3\left(\|u_t\|^2+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|y^2(x,1,\varsigma,t)\,d\varsigma dx\right) \\ &+c_3\left(\|v_t\|^2+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|z^2(x,1,\varsigma,t)\,d\varsigma dx\right). \end{aligned} \tag{27}$$

Therefore

$$H'(t)\ge c_3\max\left\{\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|y^2(x,1,\varsigma,t)\,d\varsigma dx,\ \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|z^2(x,1,\varsigma,t)\,d\varsigma dx\right\}\ge 0 \tag{28}$$

and

$$\begin{aligned} 0\le H(0)&\le H(t) \\ &\le\frac{1}{2(p+2)}\left[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{2(p+2)}^{2(p+2)}\right] \\ &\le\frac{c_1}{2(p+2)}\left[\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right]. \end{aligned} \tag{29}$$

We set

$$K(t)=H(t)+\varepsilon\int_\Omega(uu_t+vv_t)\,dx+\frac{\varepsilon}{2}\int_\Omega\left(\mu_1u^2+\mu_3v^2\right)dx \tag{30}$$

where $\varepsilon>0$ to be given later.

We multiply the first and second equations on (15) respectively by u, v and with a derivative of (30), we obtain

$$\begin{aligned} K'(t)=&H'(t)+\varepsilon\left(\|u_t\|^2+\|v_t\|^2\right)-\varepsilon\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ &-\varepsilon\int_\Omega\|\nabla u\|^{2\gamma}|\nabla u|^2\,dx-\varepsilon\int_\Omega\|\nabla v\|^{2\gamma}|\nabla v|^2\,dx \\ &+\varepsilon\int_\Omega\nabla u\int_0^t g(t-s)\nabla u(s)\,dsdx+\varepsilon\int_\Omega\nabla v\int_0^t h(t-s)\nabla v(s)\,dsdx \\ &-\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|uy(x,1,\varsigma,t)\,d\varsigma dx-\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|vz(x,1,\varsigma,t)\,d\varsigma dx \\ &+\varepsilon\left[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{2(p+2)}^{2(p+2)}\right]. \end{aligned} \tag{31}$$

By using Young's inequality, we obtain

$$\begin{aligned}&\varepsilon\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|uy(x,1,\varsigma,t)\,d\varsigma dx\\ &\leq \varepsilon\delta_1\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|d\varsigma\right)\|u\|^2\\ &+\frac{\varepsilon}{4\delta_1}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|y^2(x,1,\varsigma,t)\,d\varsigma dx.\end{aligned}\tag{32}$$

Therefore

$$\begin{aligned}&\varepsilon\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|vz(x,1,\varsigma,t)\,d\varsigma dx\\ &\leq \varepsilon\delta_2\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|d\varsigma\right)\|v\|^2\\ &+\frac{\varepsilon}{4\delta_2}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|z^2(x,1,\varsigma,t)\,d\varsigma dx.\end{aligned}\tag{33}$$

Moreover,

$$\begin{aligned}&\varepsilon\int_0^t g(t-s)\,ds\int_{\Omega}\nabla u\nabla u(s)\,dxds\\ &=\varepsilon\int_0^t g(t-s)\,ds\int_{\Omega}\nabla u(\nabla u(s)-\nabla u(t))\,dxds\\ &+\varepsilon\left(\int_0^t g(s)\,ds\right)\|\nabla u\|^2\\ &\geq\frac{\varepsilon}{2}\left(\int_0^t g(s)\,ds\right)\|\nabla u\|^2-\frac{\varepsilon}{2}(go\nabla u),\end{aligned}\tag{34}$$

hence

$$\begin{aligned}&\varepsilon\int_0^t h(t-s)\,ds\int_{\Omega}\nabla v\nabla v(s)\,dxds\\ &=\varepsilon\int_0^t h(t-s)\,ds\int_{\Omega}\nabla v(\nabla v(s)-\nabla v(t))\,dxds\\ &+\varepsilon\left(\int_0^t g(s)\,ds\right)\|\nabla v\|^2\\ &\geq\frac{\varepsilon}{2}\left(\int_0^t h(s)\,ds\right)\|\nabla v\|^2-\frac{\varepsilon}{2}(go\nabla v).\end{aligned}\tag{35}$$

By (31),

$$\begin{aligned}K'(t)&\geq H'(t)+\varepsilon\left(\|u_t\|^2+\|v_t\|^2\right)-\varepsilon\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right)\\ &-\varepsilon\left(\left(1-\frac{1}{2}\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\left(1-\frac{1}{2}\int_0^t h(s)\,ds\right)\|\nabla v\|^2\right)\\ &-\varepsilon\delta_1\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|d\varsigma\right)\|u\|^2-\varepsilon\delta_2\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|d\varsigma\right)\|v\|^2\\ &-\frac{\varepsilon}{2}(go\nabla u)-\frac{\varepsilon}{4\delta_1}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|y^2(x,1,\varsigma,t)\,d\varsigma dx\\ &-\frac{\varepsilon}{2}(ho\nabla v)-\frac{\varepsilon}{4\delta_2}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|z^2(x,1,\varsigma,t)\,d\varsigma dx\\ &+\varepsilon\left[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{2(p+2)}^{2(p+2)}\right].\end{aligned}\tag{36}$$

Thus by choosing δ_1, δ_2 such that $\frac{1}{4\delta_1 c_3} = \frac{1}{4\delta_2 c_3} = \frac{\kappa}{2}$ and by using (28) in (36), we obtain

$$
\begin{aligned}
K'(t) \geq \left[1-\varepsilon\kappa\right] H'(t) + \varepsilon\left(\|u_t\|^2 + \|v_t\|^2\right) - \varepsilon\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) \\
-\varepsilon\left[\left(1-\frac{1}{2}\int_0^t g(s)\,ds\right)\right]\|\nabla u\|^2 - \varepsilon\left[\left(1-\frac{1}{2}\int_0^t h(s)\,ds\right)\right]\|\nabla v\|^2 \\
-\varepsilon\frac{1}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|\,d\varsigma\right)\|u\|^2 - \frac{\varepsilon}{2}(go\nabla u) \\
-\varepsilon\frac{1}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|\,d\varsigma\right)\|v\|^2 - \frac{\varepsilon}{2}(ho\nabla v) \\
+\varepsilon\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{2(p+2)}^{2(p+2)}\right].
\end{aligned} \tag{37}
$$

By (25) and for $0<a<1$, we have

$$
\begin{aligned}
\varepsilon\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{2(p+2)}^{2(p+2)}\right] \\
= \varepsilon a\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{2(p+2)}^{2(p+2)}\right] \\
+2\varepsilon(p+2)(1-a)H(t) \\
+\varepsilon(p+2)(1-a)\left(\|u_t\|^2 + \|v_t\|^2\right) \\
+\varepsilon\frac{(p+2)(1-a)}{(\gamma+1)}\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) \\
+\varepsilon(p+2)(1-a)\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2 \\
+\varepsilon(p+2)(1-a)\left(1-\int_0^t h(s)\,ds\right)\|\nabla v\|^2 \\
+\varepsilon(p+2)(1-a)(go\nabla u) \\
+\varepsilon(p+2)(1-a)(ho\nabla u) \\
+\varepsilon(p+2)(1-a)M(y,z).
\end{aligned} \tag{38}
$$

Substituting in (37), we obtain

$$
\begin{aligned}
K'(t) \geq & [1-\varepsilon\kappa]H'(t) + \varepsilon\left[(p+2)(1-a)+1\right]\left(\|u_t\|^2 + \|v_t\|^2\right) \\
& +\varepsilon\left[\frac{(p+2)(1-a)}{(\gamma+1)} - 1\right]\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) \\
& +\varepsilon\left[(p+2)(1-a)\left(1-\int_0^t g(s)\,ds\right) - \left(1-\frac{1}{2}\int_0^t g(s)\,ds\right)\right]\|\nabla u\|^2 \\
& +\varepsilon\left[(p+2)(1-a)\left(1-\int_0^t h(s)\,ds\right) - \left(1-\frac{1}{2}\int_0^t h(s)\,ds\right)\right]\|\nabla v\|^2 \\
& -\varepsilon\frac{1}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|\,d\varsigma\right)\|u\|^2 - \varepsilon\frac{1}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|\,d\varsigma\right)\|v\|^2 \\
& +\varepsilon(p+2)(1-a)M(y,z) + \varepsilon\left[(p+2)(1-a) - \frac{1}{2}\right]\left((go\nabla u) + (ho\nabla u)\right) \\
& +\varepsilon a\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{2(p+2)}^{2(p+2)}\right] + 2\varepsilon(p+2)(1-a)H(t).
\end{aligned}
$$

Utilizing Poincare's inequality, we get

$$
\begin{aligned}
K'(t) \geq & [1-\varepsilon\kappa]H'(t) + \varepsilon\left[(p+2)(1-a)+1\right]\left(\|u_t\|^2 + \|v_t\|^2\right) \\
& +\varepsilon\left[\frac{(p+2)(1-a)}{(\gamma+1)} - 1\right]\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) \\
& +\varepsilon\left\{\left[(p+2)(1-a)-1\right] - \left(\int_0^t g(s)\,ds\right)\left[(p+2)(1-a) - \frac{1}{2}\right]\right\}\|\nabla u\|^2 \\
& \left\{-\frac{c}{2\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varsigma)|\,d\varsigma\right)\right\}\|\nabla u\|^2 \\
& +\varepsilon\left\{\left[(p+2)(1-a)-1\right] - \left(\int_0^t g(s)\,ds\right)\left[(p+2)(1-a) - \frac{1}{2}\right]\right\}\|\nabla v\|^2 \\
& \left\{-\frac{c}{2\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varsigma)|\,d\varsigma\right)\right\}\|\nabla v\|^2 \\
& +\varepsilon(p+2)(1-a)M(y,z) + \varepsilon\left[(p+2)(1-a) - \frac{1}{2}\right]\left((go\nabla u) + (ho\nabla u)\right) \\
& +\varepsilon c_0 a\left[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right] \\
& +2\varepsilon(p+2)(1-a)H(t).
\end{aligned}
\tag{39}
$$

Here, by taking $a > 0$ small enough which gives

$\alpha_1 = (p+2)(1-a) - 1 > 0$ and $\left(\frac{(p+2)(1-a)}{(\gamma+1)} - 1\right) > 0$

also we suppose that

$$
\max\left\{\int_0^\infty g(s)\,ds,\ \int_0^\infty h(s)\,ds\right\} < \frac{(p+2)(1-a)-1}{(p+2)(1-a)-\frac{1}{2}} = \frac{\alpha_1}{\alpha_1 + \frac{1}{2}}. \tag{40}
$$

Choosing κ so large that

$$\alpha_2 = \left(\left(p+2\right)\left(1-a\right)-1\right)-\int_0^t g\left(s\right)ds\left(\left(p+2\right)\left(1-a\right)-\frac{1}{2}\right)$$
$$-\frac{c}{2\kappa}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|ds\right)$$
$$>0,$$
$$\alpha_3 = \left(\left(p+2\right)\left(1-a\right)-1\right)-\int_0^t h\left(s\right)ds\left(\left(p+2\right)\left(1-a\right)-\frac{1}{2}\right)$$
$$-\frac{c}{2\kappa}\left(\int_{\tau_1}^{\tau_2}\left|\mu_4\left(s\right)\right|ds\right)$$
$$>0.$$

After fixing κ and $a,$ we assign ε small enough such that

$$\alpha_4 = 1-\varepsilon\kappa > 0,$$

and

$$K\left(t\right)\le \frac{c_1}{2\left(p+2\right)}\left[\left\|u\right\|_{2(p+2)}^{2(p+2)}+\left\|v\right\|_{2(p+2)}^{2(p+2)}\right]. \tag{41}$$

Therefore, for some $\beta>0,$ the estimate (39) becomes

$$\begin{aligned} K'\left(t\right)\ge{}& \beta\left\{H\left(t\right)+\left\|u_t\right\|^2+\left\|v_t\right\|^2+\left\|\nabla u\right\|^2+\left\|\nabla v\right\|^2+\left\|\nabla u\right\|^{2(\gamma+1)}+\left\|\nabla v\right\|^{2(\gamma+1)}\right\} \\ &+\beta\left\{\left(go\nabla u\right)+\left(ho\nabla v\right)+M\left(y,z\right)+\left[\left\|u\right\|_{2(p+2)}^{2(p+2)}+\left\|v\right\|_{2(p+2)}^{2(p+2)}\right]\right\}. \end{aligned} \tag{42}$$

From (11), for some $\beta_1>0,$

$$\begin{aligned} K'\left(t\right)\ge{}& \beta_1\left\{H\left(t\right)+\left\|u_t\right\|^2+\left\|v_t\right\|^2+\left\|\nabla u\right\|^2+\left\|\nabla v\right\|^2+\left\|\nabla u\right\|^{2(\gamma+1)}+\left\|\nabla v\right\|^{2(\gamma+1)}\right\} \\ &+\beta_1\left\{\left(go\nabla u\right)+\left(ho\nabla v\right)+M\left(y,z\right)+\left[\left\|u+v\right\|_{2(p+2)}^{2(p+2)}+2\left\|uv\right\|_{2(p+2)}^{2(p+2)}\right]\right\} \end{aligned} \tag{43}$$

and

$$K\left(t\right)\ge K\left(0\right)>0,\ t>0. \tag{44}$$

Now, we use Young's and Poincare's inequalities, hence from (30), we obtain

$$\begin{aligned} K\left(t\right)={}&\left(H\left(t\right)+\varepsilon\int_\Omega\left(uu_t+vv_t\right)dx+\frac{\varepsilon}{2}\int_\Omega\left(\mu_1u^2+\mu_3v^2\right)dx\right) \\ \le{}&\left\{H\left(t\right)+\left|\int_\Omega\left(uu_t+vv_t\right)dx\right|+\left\|\nabla u\right\|^2+\left\|\nabla v\right\|^2\right\} \\ \le{}& c\left[H\left(t\right)+\left\|u_t\right\|^2+\left\|v_t\right\|^2+\left\|\nabla u\right\|^2+\left\|\nabla v\right\|^2+\left\|\nabla u\right\|^{2(\gamma+1)}+\left\|\nabla v\right\|^{2(\gamma+1)}\right] \\ \le{}& c\left[H\left(t\right)+\left\|u_t\right\|^2+\left\|v_t\right\|^2+\left\|\nabla u\right\|^2+\left\|\nabla v\right\|^2+\left\|\nabla u\right\|^{2(\gamma+1)}+\left\|\nabla v\right\|^{2(\gamma+1)}\right] \\ &+\left[\left(go\nabla u\right)+\left(ho\nabla v\right)+\left\|u\right\|_{2(p+2)}^{2(p+2)}+\left\|v\right\|_{2(p+2)}^{2(p+2)}\right] \end{aligned} \tag{45}$$

for some $c>0.$ Utilizing the inequalities in (42) and (45) we get the differential inequality

$$K'\left(t\right)\ge\lambda K\left(t\right), \tag{46}$$

where $\lambda>0,$ depending only on β and $c.$

A simple integration of (46) gives us

$$K\left(t\right)\ge K\left(0\right)e^{\lambda t}\ \text{ for any } t>0. \tag{47}$$

From (41) and (47), we obtain

$$\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \geq Ce^{\lambda t}, \ \forall t > 0.$$

Hence, we completed the proof.

CONCLUSION

In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there was no growth of solutions for the coupled viscoelastic Kirchhoff system with distributed delay terms. We have been proved the growth of solutions for problem (1) under the sufficient conditions. This improves and extends many results in the literature.

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