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On the Diophantine Equation $x^2 + 2^a.7^b = y^n$

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ABSTRACT

Diophantine equation is known as a polynomial equation with two or more unknowns which only integral solutions are sought. In this paper, we concentrate on finding an integral solutions to the Diophantine equation $x^2 + 2^a.7^b = y^n$ for $(a,b) = (6,3)$ and $n = 3$. From this study, we found that the solutions to the equation are $(x,y) = (104,32), (392,56), (1176,112)$ and $(15288,616)$.

Keywords: Diophantine equation, Exponential Diophantine Equation, Parity

INTRODUCTION

The Diophantine equations that related or similar with the form of $x^2 + 2^a.7^b = y^n$ where n is an odd prime has been studied in some recent paper for a certain value of x and y. By considering some cases, they found that there are infinitely many solutions to the equation. (Refik, 2018) proved the Diophantine equation $(a^n - 1)(b^n - 1) = x^2$ and found that there is no solution to this equation when $n > 4$ for the case n is even number and proceed for the case $(a,b) = (2,50), (4,49), (12,45), (13,76), (20,77), (28,49)$ and $(45,100)$. From this study, he found that there is no integral solution to the equation for b is even. (I. Naci and G. Soydan, 2013) proved that the Diophantine equation $x^2 + 2^a.3^b = 11^c$ for case $a,b,c,x,y,n > 3$ and where x is coprime. They found that this equation has many integral solution for $n = 3,4,5,6,10$. Then, (S.Gou and Xi'an, 2012) find all solution to the Diophantine equation $x^2 + 2^a.17^b = 11^c$ for values $n \geq 3, a,b \geq 0$ and $a,b \in Z$ and the solution to this equation are $(x,y,n,a,b) = (5,3,3,1,0), (7,3,4,5,0), (11,5,3,2,0), (8,3,4,0,1), (1087,33,4,8,1), (5,7,4,7,1), (9,5,4,5,1)$ $(47,9,4,8,1), (47,3,8,8,1)$ and $(495,23,4,11,1)$. (G.Soydan and H.L.Zhu, 2012) extend the equation in the form of $x^2 + 2^a.19^b = y^n$. They solved for the case $n = 3,4,5$ and found many solution but they found there is no solution for the case $n > 5$. Next, a-s studied the equation in the form of $x^2 + 5^a.11^b = y^n$. for the case $gcd(x,y) = 1$ and $n > 3$. They found a unique solution when $n = 6$. That is the only integer solution is $(a,b,x,y) = (1,1,3,2)$. If $n = 5$ or $n > 7$, there is no integer solution for (a,b,x,y).

MAIN SECTION

In this section, we discussed on finding an integral solutions to the Diophantin equation $x^2 + 2^a.7^b = y^n$ for $n = 3$ and $(a,b) = (6,3)$. In order to solve this equation, we will consider two cases for the parity of x and y. By looking at the pattern of the solution and considering some cases, we obtain the following result.

Firstly, we consider for the parity of x and y both are even integers.

Theorem 1 Let be a, b, x, y, n be positive integers, then an integral solution to Diophantine equation $x^2 + 2^a.7^b = y^n$ for $(a, b, n) = (6,3,3)$ are $(x, y) =$ $(104, 32), (392,56), (1176,112)$ and $(15288,616)$.

Proof: Based on the hypothesis above, we have

Consider the equation

$$x^2 + 2^a.7^b = y^n. \quad (1)$$

From the hypothesis, (1) become

$$x^2 + 2^6.7^3 = y^3 \quad (2)$$

In order to solve this equation, we will consider a seven cases depend on the possibility of the parity of x and y.

Now, we consider the first case where both x, y are even.
Suppose $x = 2^\alpha s$ and $y = 2^\beta r$, where $(2,s) = (2,r) = 1$, $\alpha, \beta \geq 1$ and $r, s \in$ N. By substituting these values into (2), we obtain,

$$2^{3\beta}r^3 - 2^{2\alpha}s^2 = 2^6.7^3 \quad (3)$$

From (3), we consider six possibilities for the case α and β as in the table below:

Table 1: Possible cases for α, β when $\alpha, \beta > 0$

(1)	$\alpha > \beta$	$\beta = 1$
(2)	$\alpha > \beta$	$\beta > 1$
(3)	$\beta > \alpha$	$1 < \beta < 4$
(4)	$\beta > \alpha$	$\beta = 4$
(5)	$\beta > \alpha$	$\beta = 5$
(6)	$\beta > \alpha, \alpha \geq 1$	$\beta > 5$
(7)	$\beta = \alpha$	$\alpha, \beta > 0$

Case (1) : Consider $\beta > \alpha$. Suppose $\beta = 1$. From equation (3), it becomes,

$$2^3r^3 - 2^{2\alpha}s^2 = 2^6.7^3$$

By simplifying the above equation, we have

$$r^3 - 2^{2\alpha-3}s^2 = 2^3.7^3$$

It is contradiction since $(2, r) = (2, s) = 1$ and LHS is odd while RHS is even.

Case (2). Consider $\alpha > \beta$ for $\beta > 1$ and equation (3) becomes,

$$2^{3\beta-6}(r^3 - 2^{2\alpha-3\beta}s^2) = 7^3$$

The equation above is contradicting since (2; r) = (2; s) = 1 and LHS is even while RHS is odd and also $\beta > 1$.

Case (3). Consider $\beta > \alpha$ for $1 < \beta < 4$ and equation (3) becomes,

$$2^{2\alpha}(2^{3\beta-2\alpha}r^3 - s^2) = 2^6.7^3.$$

It is contradiction since LHS is odd while RHS is even for all possibilities values of α and β.

Case (4). Consider $\beta > \alpha$. Suppose $\beta = 4$. From equation (3), it becomes,

$$2^{12}r^3 - 2^{2\alpha}s^2 = 2^6.7^3. \quad (4)$$

Since $\beta > \alpha,$ the least value of α is 3. By substituting these values into (4), we obtain

$$2^6r^3 - s^2 = 7^3\,. \quad (5)$$

Since RHS=LHS has factor of 7, therefore equation (5) have a solution in the form of $s = 7w_1$ and $r = 7w_2$. Substitute these value into (5), we have

$$64(7w_2)^3 - (7w_1)^2 = 7^3\,. \quad (6)$$

From equation (6), then we have equation,

$$(64{w_2}^3 - 1) = {w_1}^2\,. \quad (7)$$

Since RHS is a square then the above equation have a solution if LHS also in the form of a square number. Therefore,

$$7|64{w_2}^3 - 1..$$

It can be written as

$$64{w_2}^3 - 1 \equiv 0 \text{ (mod 7)}.$$

That is

$${w_2}^3 \equiv 1 \text{ (mod 7)}.$$

Then, factorized the above equation, we obtain

$$(w_2 - 1)(\,{w_2}^2 + w_2 + 1) \equiv 0 \text{ (mod 7)}. \quad (8)$$

By solving the equation above we get w2 = 1 in the least residue modulo 7, then substitute in (7) we have

$$7(63) = {w_1}^2\,.$$

That is, $w_1 = 21$. Thus, we have $s = 147$ and r = 7. By substituting these value into x and y with $\alpha = 3$ and $\beta = 4$, we obtain

$$x = 1176, y = 112$$

If ${w_2}^2 + w_2 + 1 \equiv 0(\text{mod } 7)$, by completing square, it can be written as

$$(\frac{2w_2 + 1}{2})^2 + \frac{3}{4} \equiv 0(\text{mod } 7).$$

Then,

$$(2w_2 + 1)^2 \equiv 4(\text{mod } 7).$$

Let $e = 2w_2 + 1$, then

$$e \equiv 2(\text{mod } 7), e \equiv -2(\text{mod } 7)$$

Suppose $e \equiv 2(\text{mod } 7)$, it can be written as, then

$$e = 2 + 7t$$

$$t = \frac{2w_2 - 1}{7}, w, t > 0$$

Suppose $w_2 = 1$, then $t = \frac{1}{7}$ and it is contradict since $t > 0$ and by back substitution, it is contradict. So, there is no solution if ${w_2}^2 + w_2 + 1 \equiv 0(\text{mod } 7)$.

Case (5). Now from Table 1, we consider for the case $\beta > \alpha$ and $\beta = 5$. . From equation (3), we have

$$2^{2\alpha}(2^{15-2\alpha}r^3 - s^2) = 2^6.7^3.$$

By comparing both sides and since LHS=RHS, we obtain $\alpha = 3$, and

$$2^9r^3 - s^2 = 7^3 \quad (9)$$

That is,

$$s^2 = -7^3 + 2^9r^3$$

It can be written as,

$$s^2 \equiv 169(\text{mod } 512).$$

By solving the congruence equation above, we have

$$s = 13; s = 13.$$

Suppose $s = 13 + 512k$, and substitute in (9), we obtain

$$512r^3 = (13 + 512k)^2 + 343.$$

By expanding and simplifying the equation above, we get

$$r^3 = 1 + 2(13k + 256k^2).$$

That is,

$$r^3 \equiv 1(\text{mod } 2).$$

Then,

$$(r - 1)(r^2 + r + 1) \equiv 0 \ (\text{mod } 2).$$

By solving the congruence equation above, we obtain $r = 1$ in the least modulo 2 and by back substitution for all values of α, β, r and s we have

$$x = 104, y = 32.$$

If $r^2 + r + 1 \equiv 0 \ (\text{mod } 2)$.by completing square, it can be written as

$$(\frac{2r+1}{2})^2 + \frac{3}{4} \equiv 0(\text{mod } 2).$$

Then,

$$(2r + 1)^2 \equiv 4(\text{mod } 2).$$

Let $e = 2r + 1$, then

$$e \equiv 2(\text{mod } 2), e \equiv 5(\text{mod } 2)$$

Suppose $e \equiv 2(\text{mod } 2)$, it can be written as, then

$$e = 2 + 2t, t \geq$$

$$t = \frac{2r-1}{2}, r, t \geq 0$$

Suppose $r = 1$, then $t = \frac{1}{2}$ and it is contradict since $t > 0$ and by back substitution, it is contradict. So, there is no solution if $r^2 + r + 1 \equiv 0(\text{mod } 2)$. Suppose $s = 499$ in the least residue 512. By using the same argument, we will obtain the same answer for x and y.

Case (6). Now from Table 1, we consider for the case $\beta > \alpha, \alpha \geq 1$ and $\beta > 5$. . From equation (3), we have

$$2^{3\beta} r^3 - 2^2 s^2 = 2^6.7^3.$$

It can be written as,

It is contradiction since LHS is odd while RHS is even for all possibilities values of α and β.

$$2^{3\beta-2} r^3 - 2^2 s^2 = 2^4.7^3.$$

Then, contradiction occurs since $(2, r) = (2, s) = 1$ and LHS is odd while RHS is even.
Case (7). Consider $\alpha = \beta$ for $\alpha, \beta > 0$. Then equation (3) becomes

$$2^{2\alpha}(2^{\alpha} r^3 - s^2) = 2^6.7^3.$$

By comparing both sides and since LHS=RHS, the above equation holds if $\alpha = 3$ and it can be written as,

$$8r^3 - s^2 = 7^3. \quad (10)$$

Since RHS=LHS and have factor of 7, therefore equation (10) have a solution in the form of $s = 7w_1$ and $r = 7w_2$. Substitute these values in (10), we have

$$8(7w_2)^3 - (7w_1)^2 = 7^3. \quad (11)$$

From equation (11), and simplify the equation, we obtain

$$7(8{w_2}^3 - 1) = {w_1}^2. \quad (12)$$

The above equation have a solution if RHS is in the form of square number. Therefore,

$$7|8{w_2}^3 - 1.$$

It can be written as,

$${w_2}^3 - 1. \equiv 0 (\text{mod } 7)$$

By factoring the equation above, we obtain

$$(w_2 - 1)({w_2}^2 + w_2 + 1) \equiv 0 \ (\text{mod } 7). \quad (13)$$

We will consider two cases. The first case is when ${w_2}^3 - 1. \equiv 0(\text{mod } 7)$. That is $w2 = 1 + 7t$, $t \in Z$. Then we choose $w2 = 1$ in the least residue modulo 7 and substitute in (12), we will have $w1 = 7$. By back substitution to all value of α, β,r and s, we have

$$x = 392, y = 56.$$

For the second case, by completing the square and simplifying the equation, we get

$$(\frac{2w_2 + 1}{2})^2 + \frac{3}{4} \equiv 0(\text{mod } 7).$$

Then,

$$(2w_2 + 1)^2 \equiv 4(\text{mod } 7).$$

Let $e = 2w_2 + 1$, then

$$e \equiv 2(\text{mod } 7), e \equiv 5(\text{mod } 7)$$

Suppose $e \equiv 2(\text{mod } 7)$, it can be written as, then

$$e = 2 + 7t, t \geq 0$$

Then,

$$t = \frac{2w_2 - 1}{7}, w, t > 0$$

The smallest positive value of w2 such that the equation has solution is $w2 = 11$ and $t = 3$. Then by back substitution for all values of α, β,r and s, we have

$$x = 15288, y = 616.$$

Suppose $e \equiv 5(\text{mod } 7)$, by using the same argument, we will obtain the same answer for x and y. Therefore, from all cases the solutions are $(a, b, n) = (6,3,3)$ are $(x, y) = (104, 32), (392,56), (1176,112)$ and $(15288,616)$. ■

Secondly, we consider for the parity of *x* and *y* both are odd integers.

Theorem 2 Let be a, b, x, y, n be positive integers, there is no integral solution to Diophantine equation $x^2 + 2^a.7^b = y^n$ for $(a, b, n) = (6,3,3)$.

Proof: Based on the hypothesis above, we have

Suppose $x = 2^\gamma k + 1$ and $y = 2^\delta j + 1$, with (2,*k*)=1 and (2,*j*) = 1 where $\gamma \geq 1$, $\delta \geq 1$ and $k, j \in \mathbb{N}$.
From (3), we have

$$(2^\delta j + 1)^3 - (2^\gamma k + 1)^2 = 2^6.7^3. \qquad (15)$$

In order to solve (15), we will consider the possibilities of γ and δ. That is, either $\gamma=\delta$, $\gamma>\delta$ or $\gamma<\delta$.

Now, we consider the first case where $\gamma=\delta=1$. Then substitute these values in (24), we have

$$(2j+1)^3-(2k+1)^2=2^6.7^3$$

By expanding and simplifying the equation above, we obtain

$$2^2j^3+3\cdot 2j^2+3j-2k^2-2k=2^5\cdot 7^3$$

Since $(2,\mathrm{k}),(2,\mathrm{j})\ =1$, then we obtain LHS is odd and RHS is even. Thus, contradiction occurs. That is, LHS ≠ RHS. By using the same method and arguments, contradiction also occurs for the case $\gamma\ >\delta$ and $\gamma<\delta$. ■

CONCLUSION

From this study, we found that the integral solution for positive integers *x* and *y* to the equation
$x^2 + 2^a.7^b = y^n$ are $(a,b,n)=(6,3,3)$ are (x,y) $(104,32),(392,56),(1176,112)$ and $(15288,616)$.

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