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The Zero Product Probability of Some Finite Rings

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ABSTRACT

The development of theory of rings was stimulated by the abundant amount of seemingly unapproachable problems. One such problem is to determine the zero-divisors of commutative and non-commutative rings. In response to this problem, the focus of this study is given to the determination of the probability of elements that have product zero of commutative and non-commutative rings. The rings being considered are the ring of integers modulo 20, the direct product of the ring of integers modulo two with ring of integers modulo nine, the ring of 2 x 2 matrices over integers modulo two and the direct product of the ring of integers modulo two with the ring of 2 x 2 matrices over integers modulo two. The zero-divisors of each ring are found first. Then, the probability of a pair of elements in each ring has product zero is found by using the definition.

Keywords: Ring, Zero Divisor, Probability

INTRODUCTION

The study of ring theory has received a special attention. Perhaps, this is mainly due to their applications in mathematics and technologies such as in the field of cryptography, supersymmetric field theories and quantum mechanics. A non-empty set R is said to be a ring when it is an abelian group with respect to addition, a semigroup with respect to multiplication and satisfies distributive laws connecting the two binary operations (Cohn, 2001). The set of integers, $\mathbb{Z}$ is taken as a prototype for a ring and frequently used as a part of motivation and example in the application of ring since most axioms defining a ring are derived from some important properties of $\mathbb{Z}$.

The concepts of probability theory are discovered mostly in group theory and ring theory. This includes the commutativity degree that has been studied in finite group and is used to determine the abelianness of the groups. Other methods of proof are considered in the problem of commutativity in finite rings. Various researches have been done to investigate the properties and concepts of commutativity in finite rings such as the zero divisors of elements in rings.

This paper consists of four sections. The first section is the introduction, followed by some basic concepts on ring theory and probability theory. Then, results and discussions are included in the third section. In the section, the probability that two elements in commutative rings and non-commutative rings have product zero are found. Finally, in the last section, all the results obtained are summarized.

SOME BASIC CONCEPTS ON RING THEORY AND PROBABILITY THEORY

The research on ring theory and probability theory has been widely done by various researchers. In this section, some definitions and properties related to the main topic namely the ring and probability theory are presented.

Definition 1 (Cohn, 2001) Ring
A structure $(R,+,\cdot)$ is a ring if R is a non-empty set with two binary operations, + (addition) and $\cdot$ (multiplication) and the following axioms are satisfied.

(i). addition in *R* is closed : $a+b\in R$
(ii). addition in *R* is associative : $(a+b)+c=a+(b+c)$ for all $a,b,c\in R$
(iii). *R* has an additive identity : $a+0=a=0=0+a$
(iv). every element of *R* has an additive inverse : $a+(-a)=0=(-a)+a$
(v). addition in *R* is commutative : $a+b=b+a$
(vi). multiplication in *R* is closed : $a\cdot b\in R$
(vii). multiplication in *R* is associative : $(a\cdot b)\cdot c=a\cdot(b\cdot c)$ for all $a,b,c\in R$
(viii). left and right distributive laws hold in *R* : $(a+b)\cdot c=a\cdot c+b\cdot c$

From (i) to (v), it can be seen that *R* is a commutative group under addition. This type of group is usually called Abelian (rather than commutative) in honour of the distinguished Norwegian mathematician N. H. Abel who investigated a class of algebraic equation related to commutative groups (Hartley and Hawkes, 1970). A semigroup is a set *S* with binary operation of multiplication satisfying the associative law (Hartley and Hawkes, 1970). As in (viii), a ring is a set *R* equipped with binary operations which are connected by distributive law.

Definition 2 (Fraleigh and Katz, 2003) Commutative Ring
A ring in which the multiplication is commutative is a commutative ring.

For instance, the set of integers modulo *n* and real numbers are commutative rings. In this study, few commutative rings are presented which include integer modulo 20 and the direct product of integer modulo two and integer modulo nine. The ring $\mathbb{Z}_{20}$ is the ring of integers modulo 20 that has 20 elements which are listed as follows :

$$\mathbb{Z}_{20} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19\}.$$

Meanwhile, the direct product of the ring of integers modulo two with ring of integers modulo nine is denoted as $\mathbb{Z}_2 \oplus \mathbb{Z}_9$. The ring $\mathbb{Z}_2 \oplus \mathbb{Z}_9$ has 18 elements which are:

$$\mathbb{Z}_2 \oplus \mathbb{Z}_9 = \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(1,0),(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8)\}.$$

If a ring does not satisfy the commutativity for multiplication, it is said to be a noncommutative ring. The non-commutative rings which are the ring of 2 x 2 matrices over integers modulo two, $M_2(\mathbb{Z}_2)$ and the direct product of the ring of integers modulo two with the ring of 2 x 2 matrices over integers modulo two, $\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)$ are discussed in this research.

If two matrices are both *n* × *n*, then their sum and product are also *n* × *n*. This shows that the ring $M_2(\mathbb{Z}_2)$ is closed under addition and multiplication. Addition and multiplication of $M_2(\mathbb{Z}_2)$ are also associative. The commutativity of addition are satisfied since for any matrices, *A* and *B* in $M_2(\mathbb{Z}_2)$, *A* + *B* is equal to *B* + *A*. In addition, $M_2(\mathbb{Z}_2)$ is a ring with additive identity of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and multiplicative identity of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Note that the left and right distributive law holds in $M_2(\mathbb{Z}_2)$. However, it can be seen that $M_2(\mathbb{Z}_2)$ is a non-commutative ring since $A \cdot B \neq B \cdot A$ for all $A, B \in M_2(\mathbb{Z}_2)$. This can be seen in the following example:

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$,

$$A \cdot B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Hence $A \cdot B \neq B \cdot A$.

Since $M_2(\mathbb{Z}_2)$ is a non-commutative ring, $\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)$ is also a non-commutative ring.

Definition 3 (Fraleigh and Katz, 2003) Zero Divisor
If *a* and *b* are two nonzero elements of a ring *R* such that *ab* = *0*, then *a* and *b* are divisors of zero. The set of all zero divisors in *R* is denoted as *Z(R)*.

In recent years, researchers are interested in finding various types of probabilities related to groups and rings. One of it is the commutativity degree of a group which is defined as the probability that two randomly chosen element in a group commute. In 1973, Gustafson considered the problem of finding probability Pr*(G)* that two elements of a finite group *G* selected at random commute. The author mentioned that the probability is less than or equal to $\frac{5}{8}$. In the following year, MacHale (1974) used the concept of commutativity degree for finite group. The author investigated the commutativity of a non-commutative group and showed that $\Pr(G) \le \frac{5}{8}$.

Commutativity degree is also being used in rings. MacHale considered the problem of commutativity degree for finite rings in 1976. Since the concept of conjugacy in groups has no obvious analogue in rings, the methods of proof are different from those Gustafson discovered (MacHale, 1974). Besides that, in 2018, Khasraw determined the probability that two elements in a ring have product zero. The author investigated the bounds of this probability of the ring of integers modulo *n*, a finite commutative ring with identity 1. The probability that two elements in a ring have product zero is defined in the following:

Definition 4 (Khasraw, 2018)
The probability *P(R)* that two elements chosen at random (with replacement) from a ring, *R* have product zero is denoted as:

$$\mathrm{P}(R) = \frac{|Ann|}{|R \times R|}$$

where $Ann = \{(x, y) \in R \times R \mid xy = 0\}$.

Clearly, $(0,0), (x,0)$ and $(0, y)$ is in Ann. Futhermore, if $xy = 0$, then *x* and *y* are the zero divisors of *R*.

RESULTS AND DISCUSSIONS

In this section, the probabilities of some rings are discussed. The commutative rings that are being considered in this study are the ring of integers modulo 20, $\mathbb{Z}_{20}$ and the direct product of the ring of integers modulo two with ring of integers modulo nine, denoted as $\mathbb{Z}_2 \oplus \mathbb{Z}_9$. On the other hand, the non-commutative rings discussed are the ring of 2 x 2 matrices over integers modulo two, $M_2(\mathbb{Z}_2)$ and the direct product of the ring of integers modulo two with the ring of 2 x 2 matrices over integers modulo two, $\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)$. The probability of two different elements in a ring that have product zero is calculated for each ring.

The Probability That Two Elements in Commutative Rings Have Product Zero

In this subsection, the probability that a random pair of elements in a ring have product zero, which from now on called as the zero product probability, is calculated for commutative rings by using Definition 3 and Definition 4. The theorems and proofs are discussed below.

Theorem 1 Let $\mathbb{Z}_{20}$ be the ring of integers modulo 20. The ring $\mathbb{Z}_{20}$ has 11 zero divisors and the zero product probability, $\mathrm{P}(\mathbb{Z}_{20})=\frac{71}{400}$.

Proof Recall from Definition 3, if the product of any two nonzero elements is equal to zero, it is called the zero divisors of the ring. Since $\mathbb{Z}_{20}$ is commutative, only either left or right zero divisors are needed to be found. It is found that the set of zero divisors of $\mathbb{Z}_{20}$, $Z(\mathbb{Z}_{20})=\left\{2,4,5,6,8,10,12,14,15,16,18\right\}$. *Ann* can be found by using the zero divisors of $\mathbb{Z}_{20}$, together with the zero element. Next, we do the counting. By Definition 4, for $x=0$, we get 20 elements meanwhile for $y=0$, we get another 19 elements excluding $(0,0)$. To get the rest of the elements in *Ann*, we choose $(x,y)\in\mathbb{Z}_{20}$ such that $xy=0$. For example, $(2,10)\in Ann$ since $2\cdot 10=0\in\mathbb{Z}_{20}$. Note that $(y,x)\in\mathbb{Z}_{20}$ since $\mathbb{Z}_{20}$ is abelian. Thus, by using Definition 4, since $|R|=|\mathbb{Z}_{20}|=20$ and $|Ann|=71$, the zero product probability, $\mathrm{P}(\mathbb{Z}_{20})=\frac{71}{20\times 20}=\frac{71}{400}$.

Theorem 2 Let $\mathbb{Z}_2\oplus\mathbb{Z}_9$ be the set of ring. The ring $\mathbb{Z}_2\oplus\mathbb{Z}_9$ has 11 zero divisors and the zero product probability, $\mathrm{P}(\mathbb{Z}_2\oplus\mathbb{Z}_9)=\frac{61}{324}$.

Proof Similar as before, only either left or right zero divisors are needed to be found since $\mathbb{Z}_2\oplus\mathbb{Z}_9$ is a commutative ring. The set of zero divisors of $\mathbb{Z}_2\oplus\mathbb{Z}_9$, $Z(\mathbb{Z}_2\oplus\mathbb{Z}_9)=\left\{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(1,0),(1,3),(1,6)\right\}$. By Definition 4, since $|R|=|\mathbb{Z}_2\oplus\mathbb{Z}_9|=18$ and $|Ann|=61$, $\mathrm{P}(\mathbb{Z}_2\oplus\mathbb{Z}_9)=\frac{61}{18\times 18}=\frac{61}{324}$.

The Probability That Two Elements in Non-Commutative Rings Have Product Zero

In this second subsection, the zero product probability is calculated for two non-commutative rings mentioned earlier by using Definition 3 and Definition 4.

Theorem 3 The ring $M_2(\mathbb{Z}_2)$ is a set of 2 x 2 matrices that has nine zero divisors and the zero product probability, $\mathrm{P}(M_2(\mathbb{Z}_2)) = \dfrac{55}{256}$.

Proof As mentioned earlier, $M_2(\mathbb{Z}_2)$ is a non-commutative ring. Therefore, the elements of $M_2(\mathbb{Z}_2)$ does not satisfy the commutativity for multiplication. By using Definition 3, both left and right zero divisors are found. The set of zero divisors of $M_2(\mathbb{Z}_2)$ is given as follows:

$$Z(M_2(\mathbb{Z}_2)) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

It is found that there are 55 possibilities that two elements in the ring have product zero. Therefore, $\mathrm{P}(M_2(\mathbb{Z}_2)) = \dfrac{55}{16 \times 16} = \dfrac{55}{256}$.

Theorem 4 The ring $\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)$ has 18 zero divisors and the zero product probability, $\mathrm{P}(\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)) = \dfrac{111}{1024}$.

Proof Since $\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)$ is a non-commutative ring, both left and right zero divisors have to be found first by using Definition 3. Since the zero divisors of 2 by 2 matrices over $\mathbb{Z}_2$ have been found in the previous example, the zero divisors can be used in this example too. The set of zero divisors of $\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)$ is given in the following.

$$Z(\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)) = \left\{ \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right), \right.$$

$$\left(0, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right), \left(1, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right),$$

$$\left(1,\begin{bmatrix}0&0\\1&0\end{bmatrix}\right),\left(1,\begin{bmatrix}0&0\\1&1\end{bmatrix}\right),\left(1,\begin{bmatrix}0&1\\0&0\end{bmatrix}\right),\left(1,\begin{bmatrix}0&1\\0&1\end{bmatrix}\right),\left(1,\begin{bmatrix}1&0\\0&0\end{bmatrix}\right),$$
$$\left.\left(1,\begin{bmatrix}1&0\\1&0\end{bmatrix}\right),\left(1,\begin{bmatrix}1&1\\0&0\end{bmatrix}\right),\left(1,\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\right\}.$$

The probability that a pair of elements in $\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)$ have product zero, $\mathrm{P}(\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)) = \frac{111}{32\times 32} = \frac{111}{1024}$.

CONCLUSION

In this research, the probability of two elements in commutative rings, namely the ring of integers modulo 20 and the direct product of ring of integers modulo two and the ring of integers modulo nine have a product zero are determined using definitions. In addition, the probability of two elements in non-commutative rings, the ring of 2 x 2 matrices over integers modulo two and the direct product of the ring of integers modulo two with the ring of 2 x 2 matrices have a product zero are also calculated. It is found that $\mathrm{P}(\mathbb{Z}_{20}) = \frac{71}{400}$, $\mathrm{P}(\mathbb{Z}_2 \oplus \mathbb{Z}_9) = \frac{61}{324}$, $\mathrm{P}(M_2(\mathbb{Z}_2)) = \frac{55}{256}$ and $\mathrm{P}(\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)) = \frac{111}{1024}$. The findings are summarized in the following table.

Table 1: The summary of findings

Type of Ring		Zero Product Probability
Commutative ring	$\mathbb{Z}_{20}$	$\frac{71}{400}$
	$\mathbb{Z}_2 \oplus \mathbb{Z}_9$	$\frac{61}{324}$
Non-commutative ring	$M_2(\mathbb{Z}_2)$	$\frac{55}{256}$
	$\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)$	$\frac{111}{1024}$

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