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Discovering Factors of Graph Polynomials

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ABSTRACT

One of the most common approaches in studying any polynomial is by looking at its factors. Over the years, different graph polynomials have been defined for both undirected and directed graphs, including the Tutte polynomial, chromatic polynomial, greedoid polynomial and cover polynomial. We consider two graph polynomials, one for undirected graphs and one for directed graphs. We first give an overview of these two polynomials. We then discuss the factors of these polynomials as well as the information that are encapsulated by these factors.

Keywords: polynomial, Tutte polynomial, greedoid polynomial, directed graph, undirected graph, factor

INTRODUCTION AND DEFINITIONS

Graph theory is a useful tool in studying the relationships among a set of objects. It is widely used in many different disciplines including mathematics, information technology, engineering, chemistry and sociology. It can be used to model many aspects in our daily life.

A graph is a representation of a set of objects where certain pairs of objects are linked. The objects are called *vertices* and the links between the objects are called *edges*. For *undirected graphs* (*graphs*), no direction is assigned on the edges. For *directed graphs* (*digraphs*), the edges are directed from one vertex to another.

Many graph polynomials have been defined for both undirected and directed graphs. Some well known polynomials of undirected graphs include the Tutte polynomial, chromatic polynomial and flow polynomial. For directed graphs, the greedoid polynomial, cover polynomial and drop polynomial are studied. One important aspect in studying any polynomial is to investigate if the polynomial can be factorised. For some graph polynomials, its factors give some insight about the structures of the associated graphs. However, the situation is sometimes more complex . We discuss two graph polynomials, the Tutte polynomial for undirected graphs and the greedoid polynomial for directed graphs. Note that the greedoid polynomial is an analogous of the Tutte polynomial.

The terminology used is mostly standard. We follow the notation and terminology as in Diestel (2010). Let $G(V,E)$ be a graph where V and E represent the vertex set and the edge set of G, respectively. The number of vertices and the number of edges of G are denoted by $|V(G)|$ and $|E(G)|$, respectively. The number of connected components of G is denoted by $k(G)$. Let $u,v \in V(G)$. If there exists an edge between u and v, the edge is *incident* to both u and v, which is written as uv. In the context of digraphs, uv represents an edge that is directed from u to v. Hence, $uv \neq vu$. An edge is a *coloop* (or *bridge*) if it is not in any cycle.

A graph H is a *subgraph* of G if H is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is a *spanning subgraph* of G.

A *cutvertex* in a graph G is a vertex v such that $k(G-v) > k(G)$. A connected graph G is *k-connected* if it contains more than $k \geq 0$ vertices and $G - X$ is connected for every $X \subseteq V(G)$ with $|X| < k$.

A *cut* is the partition of the vertex set of the graph into two disjoint subsets. A *cutset* of a cut is the set of edges that have one endpoint in each of these subsets. A *bond* is a minimal non-empty cut in a graph.

A *block* of a graph is a maximal connected subgraph that contains no cutvertex. A subgraph H of a graph G is a block if H is either a maximal 2-connected subgraph, a bridge with its two endpoints or an isolated vertex.

Let G be a graph. The *rank* $r(X)$ of a set of edges $X \subseteq E(G)$ is the number of vertices it meets minus the number of connected components of the subgraphs.

An *acyclic graph* is a graph that contains no cycles. A forest is an acyclic graph. If a forest is connected, then it is a *tree*. A *spanning tree* T of a graph G is a spanning subgraph of G that is a tree. For every edge $e \in E(G)\backslash E(T)$, there is a unique cycle in $T + e$. The unique cycle is the *fundamental cycle* of e with respect to T. For every edge $f \in E(T)$, the forest $T - f$ has exactly two components. The set of edges of G between these components is a bond in G, which is the *fundamental cutset* of f with respect to T.

Let D be a digraph with a distinguished vertex v. We call v the *root* of D, and D a digraph *rooted at* v.

An *arborescence* (Tutte, 1973) is a directed tree rooted at a vertex v such that every edge that is incident with v is an outgoing edge, and exactly one edge is directed into each of the other vertices.

Let D be a rooted digraph. A subdigraph F of D is *feasible* if F is an arborescence. We call the edge set of F a *feasible set*. The *rank* of a subset $X \subseteq E(D)$ is defined as $r(X) = \max\{|A|: A \subseteq X, A \text{ is feasible}\}$.

Let D, D_1 and D_2 be rooted digraphs, and $E(D_1), E(D_2) \subseteq E(D)$. The digraph D is the *direct sum* $D = D_1 \oplus D_2$ of D_1 and D_2, if $E(D_1) \cup E(D_2) = E(D)$, $E(D_1) \cap E(D_2) \neq \emptyset$ and the feasible sets of D are precisely the unions of the feasible sets of D_1 and D_2.

A *matroid* (Welsh, 1976; Oxley, 2011) over a finite *ground set* E is an order pair $(E, \mathcal{I})$ where $\mathcal{I} \subseteq 2^E$ is a collection of subsets of E satisfying the following three properties:

(M1) $\emptyset \notin \mathcal{I}$.

(M2) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$.

(M3) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then there is an element $y \in Y - X$ such that $X \cup \{y\} \in \mathcal{I}$.

Greedoids (Korte and Lovász, 1981) are a generalisation of matroids. A *greedoid* over a finite set E is a pair (E, F) where $F \subseteq 2^E$ is a non-empty collection of subsets of E satisfying:

(G1) For every non-empty $X \in F$, there is an element $x \in X$ such that $X - \{x\} \in F$.

(G2) For $X, Y \in F$ with $|X| < |Y|$, then there is an element $y \in Y - X$ such that $X \cup \{y\} \in F$.

THE TUTTE POLYNOMIAL

The Tutte polynomial (Tutte, 1954; Tutte, 1967; Tutte, 1974; Welsh, 1993; Welsh, 1999) is a two-variable polynomial that is defined for every undirected graph. It contains a variety of information about other polynomials including the chromatic polynomial, flow polynomial and reliability polynomial. The Tutte polynomial also specialises to the partition functions of the Potts model and Ising model in statistical physics.

The Tutte polynomial of a graph G can be defined using three different ways: by the (i) state sum expansion, (ii) deletion-contraction recurrence and (iii) notion of basis activities.

i) State sum expansion:

$$T(G;x,y) = \sum_{X\subseteq E(G)} (x-1)^{r(E)-r(X)} (y-1)^{|X|-r(X)}. \tag{1}$$

The Tutte polynomial is closely related to a two-variable polynomial, namely the *Whitney-rank generating function* (Whitney, 1932)

$$R(G;x,y) = \sum_{X\subseteq E(G)} x^{r(E)-r(X)} y^{|X|-r(X)}. \tag{2}$$

By performing coordinate transformation between (1) and (2), it is easy to see that

$$T(G;x,y) = R(G;x-1,y-1).$$

ii) *Deletion-contraction recurrence*:

$$T(G;x,y)\begin{cases} 1, & \text{if } G \text{ is empty,} \\ x\cdot T(G\backslash e;x,y), & \text{if } e \text{ is a bridge,} \\ y\cdot T(G\backslash e;x,y), & \text{if } e \text{ is a loop,} \\ T(G/e;x,y)+T(G\backslash e;x,y), & \text{otherwise.} \end{cases}$$

Note that the deletion and contraction operations commute. Hence this recurrence generates the same polynomial for any given graph, regardless of the edge-ordering.

iii) Notion of *basis activities*:

Let T be a spanning tree of G and $<$ be a total order on $E(G)$. An edge $e \in E(T)$ is *internally active* if e has the maximum order by the ordering of $<$ in the fundamental cutset in T. The *internal activity* $\text{int}(T)$ of T is the number of edges that are internally active in T. An edge $e \in E(G)\backslash E(T)$ is *externally active* if e has the maximum order by the ordering of $<$ in the fundamental cycle in T. The *external activity* $\text{ext}(T)$ of T is the number of edges that are externally active in T. Suppose $\mathcal{T}(G)$ be the set of spanning trees of G. Then

$$T(G;x,y) = \sum_{T\subseteq \mathcal{T}(G)} x^{\text{int}(T)} y^{\text{ext}(T)}. \tag{3}$$

Note that (3) is independent of the choice of $<$ (Tutte, 1954). For a graph G that has $k > 1$ components $G_1, G_2, \cdots, G_k$,

$$T(G;x,y) = \prod_{i=1}^{k} T(G_i;x,y).$$

The Tutte polynomial is also a counting function that gives a variety of information about a graph, by substituting appropriate values for both variables x and y. For instance,

- $T(G;1,1)$ counts the number of spanning trees of a connected graph G.
- $T(G;1,2)$ counts the number of spanning subgraphs of G.
- $T(G;2,1)$ counts the number of forests of G.

Example: Suppose the Tutte polynomial of the graph G that is shown in Figure 1, is determined by using the state sum expansion.

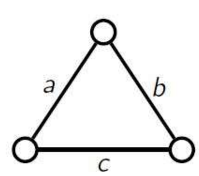

Figure 1: A graph G with the edge set $\{a, b, c\}$

Since $|E(G)| = 3$, there exist 8 possible subsets of the edges of G. By determining the rank of each subset and using (1), we can easily see that

$$T(G; x, y) = x^2 + x + y. \quad (4)$$

By substituting $x = 1$ and $y = 1$ into (4), we have $T(G; 1,1) = 3$. This implies that G contains 3 spanning trees, which are $\{a, b\}$, $\{b, c\}$ and $\{a. c\}$.

Tutte proved one interesting property about the Tutte polynomial in the following theorem (Tutte, 1954).

Theorem 1. *If a graph G consists of two connected graphs H_1 and H_2 having just one vertex in common, then*

$$T(G; x, y) = T(H_1; x, y) \cdot T(H_2; x, y).$$

An analogous result to Theorem 1 holds for matroids.

Suppose M, M_1 and M_2 are matroids. Merino et al. (2001) proved that if

$$T(M; x, y) = T(M_1; x, y) \cdot T(M; x, y),$$

then M is the direct sum of M_1 and M_2. Their proof verified a conjecture by Brylawski (1972). This concludes that the Tutte polynomial of a matroid (respectively, graph) factorises if and only if the matroid (respectively, graph) is a direct sum. Hence, factorisation of the Tutte polynomial of a graph reflects the structures of the graph.

THE GREEDOID POLYNOMIAL

Gordon and McMahon (1989) defined a two-variable greedoid polynomial

$$f(G; t, z) = \sum_{A \subseteq E(G)} t^{r(E)-r(A)} \, z^{|A|-r(A)}$$

for any greedoid G. The two-variable greedoid polynomial generalises the one-variable greedoid polynomial $\lambda(G; t)$ given by Björner and Ziegler (1992). We call the two-variable greedoid polynomial the *greedoid polynomial*. Important classes of greedoids are those associated with rooted graphs (*branching greedoids*) and rooted digraphs (*directed branching greedoids*).

Gordon and McMahon (1989) investigated greedoid polynomials for branching greedoids and directed branching greedoids. They showed that $f(D; t, z)$ can be used to determine if a rooted digraph D us a rooted arborescence. This result however does not extend to unrooted trees (Eisenstat and Gordon, 2006).

Gordon and McMahon proved that the greedoid polynomials of rooted digraphs have the multiplicative direct sum property, that is, if a digraph $D = D_1 \oplus D_2$, then

$$f(D; t, z) = f(D_1; t, z) \cdot f(D_2; t, z).$$

Note that there is an analogous result for the c-Tutte invariant of alternating dimaps (see Yow et al. (2018a) or Yow (2019)).

Let D be a rooted digraph. Gordon and McMahon (1989) introduced a recurrence formula to compute $f(D; t, z)$, which involves the usual deletion-contraction operations.

Proposition 2. *Let D be a digraph rooted at a vertex v, and e be an outgoing edge of v. Then*

$$f(D; t, z) = f(D/e; t, z) + t^{r(D)-r(D \backslash e)} f(D \backslash e; t, z).$$

A *greedoid loop* in a rooted graph or a rooted digraph is an edge that is in no feasible set. It is either an ordinary (directed) loop, or an edge that belongs to no (directed) path from the root node.

Factorisation of greedoid polynomials behaves differently compared to the Tutte polynomial. Gordon and McMahon showed that the greedoid polynomial of a rooted digraph that is not necessarily a direct sum has $1+z$ among its factors under certain conditions (McMahon, 1993; Gordon and McMahon, 1997).

Yow at el. (2018b) addressed more general types of factorisation for greedoid polynomials of rooted digraphs, by computing the greedoid polynomials for all rooted digraphs up to order six. They studied the factorability of greedoid polynomials of rooted digraphs, particularly those that are not divisible by $1+z$.

Two rooted digraphs are *GM-equivalent* if they both have the same greedoid polynomial. If a rooted digraph is a direct sum, then it is *separable*. Otherwise, it is *non-separable*.

A greedoid polynomial $f(D)$ if a rooted digraph D of order n *GM-factorises* if $f(D) = f(G) \cdot f(H)$ such that G and H are rooted digraphs of order at most n and $f(G), f(H) \neq 1$. A rooted digraph *GM-factorises* if its greedoid polynomial GM-factorises.

A *GM-factor* of a rooted digraph D is a polynomial P where P divides $f(D)$ and $P \neq 1$. An irreducible GM-factor is *basic* if the GM-factor is either $1+t$ or $1+z$. Otherwise, the irreducible GM-factor is *nonbasic*.

The numbers of various types of rooted digraphs up to order six are given in Table 1 (Yow et al., 2018b), where T represents the number of labelled rooted digraphs, T-ISO represents the number of rooted digraphs, S represents the number of separable digraphs, NS represents the number of non-separable digraphs and NSE represents the number of non-separable digraphs of order n that are GM-equivalent to some separable digraph of order at most n.

Table 1: Numbers of various types of rooted digraphs

n	T	T-ISO	S	NS	NSE
1	1	1	0	1	0
2	6	4	0	4	0
3	48	36	6	30	7
4	872	752	88	664	200
5	48040	45960	2404	43556	10641
6	9245664	9133760	150066	8983694	1453437

Yow et al. (2018b) showed that there exist non-separable rooted digraphs that can be GM-factorised. One such example is shown in Figure 2 where its greedoid polynomial is $(1+t)(1+z)$.

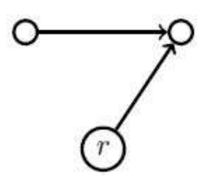

Figure 2: A non-separable rooted digraph of order 3

They also found non-separable rooted digraphs that GM-factorise and contains no basic factor. The rooted digraph D in Figure 3 is non-separable and its greedoid polynomial is

$$f(D) = (1+t+t^2+t^2z)(2+2t+t^2+t^3+z+tz+t^2z+3t^3z+3t^3z^2+t^3z^3).$$

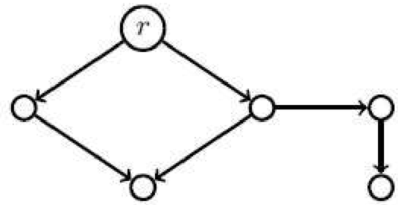

Figure 3: A non-separable rooted digraph of order 6 that contains no basic factor

Note also that both nonbasic factors $(1+t+t^2+t^2z)$ and $(2+2t+t^2+t^3+z+tz+t^2z+3t^3z+3t^3z^2+t^3z^3)$ are greedoid polynomials of rooted digraphs of smaller order. However, the rooted digraph D is non-separable and hence is not a direct sum of other rooted digraphs.

Let P_{m,v_0} be a *directed path* $v_0v_1\cdots v_m$ of size $m\geq 0$ rooted at v_0, and C_{m,v_0} be a *directed cycle* $v_0v_1\cdots v_{m-1}v_0$ of size $m\geq 1$ rooted at v_0. We usually write P_m for P_{m,v_0} and C_m for C_{m,v_0}.

Yow et al. (2018b) characterised the greedoid polynomials for P_m and C_m. They then proved that there exists an infinite family of non-separable digraphs where their greedoid polynomials factorise into at least two nonbasic GM-factors, in the following theorem.

Theorem 3. *There exists an infinite family of non-separable digraphs D that have at least two nonbasic GM-factors, where*

$$f(D)=f(P_{k+1})\left(f(C_{k+1})+f(P_{k+1})+\frac{t^{k+2}(1+z)^{k+2}\left(1-\left(t(1+z)\right)^{l}\right)}{1-t(1+z)}\right), for\ k,l\geq 1.$$

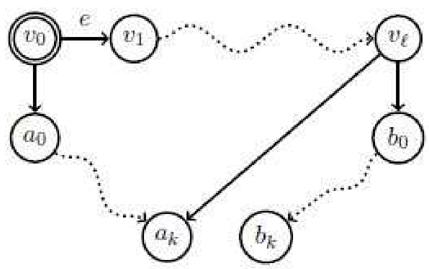

Figure 3: The digraph D in the proof of Theorem 3

They also gave an infinite family of rooted digraphs where each greedoid polynomial of these digraphs is a nonbasic GM-factor of the greedoid polynomial of some non-separable digraph.

Theorem 4. *For any digraph G that has a directed path of length at least two, there exists a non-separable digraph D where $f(D)$ has $f(G)$ as a nonbasic GM-factor.*

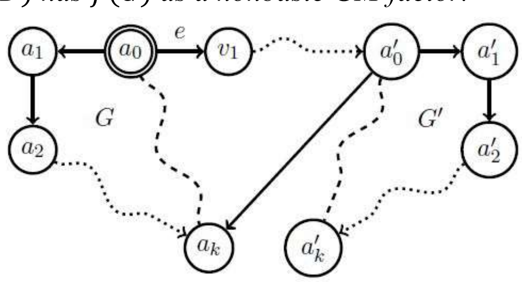

Figure 4: An illustration of the digraph D in Theorem 4 where G' is a copy of G

These findings show that factorisation of the greedoid polynomials of rooted digraphs do not reflect the structures of the associated rooted digraphs. There are some other factors that determine if the greedoid polynomials of rooted digraph can be factorised, which can be seen from the above counterexamples.

CONCLUSION

Factorisation of the greedoid polynomial of rooted digraphs is more complex compared to the factorisation of the Tutte polynomial of undirected graphs. In the former, the multiplicative direct sum property, the existence of greedoid loops and the directed cycles, are not the only characteristics that determine if greedoid polynomials of rooted digraphs can be factorised. In the latter, the Tutte polynomial of a graph factorises if and only if the graph is a direct sum. Although the greedoid polynomial is an analogous of the Tutte polynomial, more work is needed in order to characterise the factorability of greedoid polynomials of rooted digraphs, which may provide some other useful information about the associated rooted digraphs.

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