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Solving Delay Differential Equations By Adams Moulton Block Method Using Divided Difference Interpolation

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ABSTRACT

This paper will consider a block method for solving delay differential equations (DDEs) using variable step size and order. The coupled block method consists of two and three point block method in a single code presented as in the simple Adams Moulton type. The code will compute the numerical solutions at two and three new values simultaneously at each of the integration step. The approximation of the delay term is estimated using the divided difference interpolation. The P-stability and Q-stability regions are also illustrated. The numerical results for the coupled block method were superior compared to the existing block method. It is clearly shown that the code is able to produce good results for solving DDEs.

***Key words*:** block method; variable step size and order; delay differential equations.

INTRODUCTION

For many years, researchers are interested in the study of numerical treatment of DDEs in order to developed efficient numerical methods for solving DDEs. Generally, DDEs involved the evolution of the system at a certain time, depends on the state of the system at an earlier time. DDEs have been used in many applications in science and engineering.

In this paper, we considered the development of the code for solving single-delay scalar DDEs of the form

$$\begin{aligned} y'(x) &= f\left(x, y, y(x-\tau)\right),\ a \le x \le b \\ y(x) &= \phi(x),\ x \le a \end{aligned} \qquad (1)$$

where $\varphi(x)$ is the initial function, $\tau(x, y(x))$ is called the delay, $x-\tau(x, y(x))$ is called the delay argument and the value of $y(x-\tau(x, y(x)))$ is the solution of the delay term. The delay is called constant delay if it is a constant, it is called time dependent delay if the delay is function of time x and the delay is known as state dependent delay if it is a function of time x and $y(x)$.

Most of the methods used for solving DDEs are commonly adapted from the existing numerical methods for solving ODEs. Ismail et al. (2002) and Oberle and Pesch (1981), the authors developed different type of Runge-Kutta methods for solving DDEs and approximated the delay term using appropriate Hermite interpolation. Numerical methods for solving DDEs using variable step size and order algorithm have been proposed by several researchers such as Ishak et al. (2008), Jackiewicz (1987), Radzi et al. (2011) and; Suleiman and Ishak (2010). Ishak et al. (2008) proposed a two point predictor-corrector block method in divided difference form for solving DDEs. The application of two point block method can simultaneously produces two new points within a block. While the numerical methods described in Ismail et al. (2002), Jackiewicz (1987), Oberle and Pesch (1981) and; Suleiman and Ishak (2010) will only estimated the numerical solutions at one point sequentially. In Ishak et al. (2010), the authors developed two point implicit block method for solving DDEs using variable step size strategy. The delay term is calculated using six points Lagrange interpolation.

The objective of this paper is to implement a coupled block method that consists of two point and three point block methods for solving (1) using variable step size and order. The coupled block method is adapted from the code proposed by San et al. (2011) for solving first order ODEs. The propose method is expected to be suitable for solving DDEs.

FORMULATION OF THE METHOD

The coupled block method CB(6,8) proposed by San et al. (2011) consists of two point three step block method (2P3S) of order six and three point four step block method (3P4S) of order eight. The derivation of those methods can be found in San et al. (2011).

Two point three step block method (2P3S)

In Figure 1, the solutions of y_{n+1} and y_{n+2} with step size *h* are simultaneously computed in a block by considering the previous three steps with step size *2rh* and *qh*.

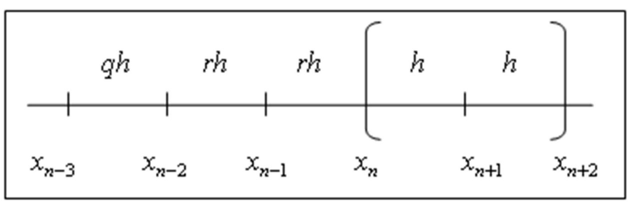

Figure 1: Two point three step block method.

The corrector formula of the two point three step block method were derived using Lagrange interpolation polynomial and the interpolation points involved are $(x_{n-3}, f_{n-3}),\ldots,(x_{n+2}, f_{n+2})$. The two values of y_{n+1} and y_{n+2} can be obtained by integrating over the interval $[x_n, x_{n+1}]$ and $[x_n, x_{n+2}]$ respectively using MAPLE and the corrector formula in terms of r and q can be obtained.

The choices for the next step size will be limited to half, double or the same as the current step size in order to minimize the storage of the formula. For example, in case of successful step size, the possible ratios for the next constant step size are (*r*=1, *q*=1), when the step size is double the possible ratio is (*r*=0.5, *q*=0.5) and in case of step size failure, the possible value is (*r*=2, *q*=2). The corrector formula will be simplified by substituting the values of *r* and *q*. The two point three step block method is the combination of predictor of order five and corrector of order six.

Three point four step block method (3P4S)

In Figure 2, the three point four step block method will compute three points simultaneously in a block by considering the previous four steps with step size *2rh, qh* and *ph*. The corrector formula of the three point four step block method in Figure 2 were derived using Lagrange interpolation polynomial and the interpolation points involved are $(x_{n-4}, f_{n-4}),\ldots,(x_{n+3}, f_{n+3})$.

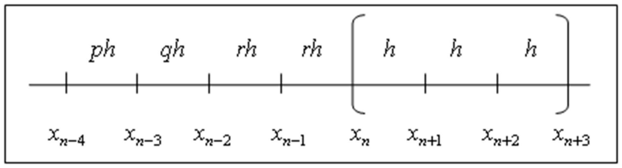

Figure 2: Three point four step block method

The three values of y_{n+1}, y_{n+2} and y_{n+3} can be obtained by integrating over the interval $[x_n, x_{n+1}], [x_n, x_{n+2}]$ and $[x_n, x_{n+3}]$ respectively and the corrector formula in terms of *r*, *q* and *p* can be obtained. The choices of *r*, *q* and *p* will varies as the step size changing to double, half or remain constant. The predictor is order seven and the corrector is order eight.

IMPLEMENTATION AND NUMERICAL TREATMENTS OF DDES

Implementation of the method

The code will be implemented in $PE(CE)^s$ mode. The *P* and *C* denote the application of predictor and corrector respectively while *E* denotes the evaluation of function *f*. The local error for two point block method at x_{n+2} can be estimated as

$$T_1 = y_{n+2}(k) - y_{n+2}(k-1) \tag{2}$$

where $y_{n+2}(k)$ is the corrector formula of order k and $y_{n+2}(k-1)$ is a similar corrector formula of order $k-1$. Similarly, we could estimate the local error for three point block method at x_{n+3} as

$$T_2 = y_{n+3}(k+2) - y_{n+3}(k+1). \tag{3}$$

Suppose that the local error test $T \le \text{TOL}$ is accepted in the integration step. The next order can be choose from one of the methods of order k and $k+2$ if the estimates step size on the next integration is the maximum. Having available T_1 and T_2, the maximum step size are as follows:

$$h_1 = h_{old} \times \left(\frac{TOL}{2.0 \times T_1} \right)^{\frac{1}{k}} \tag{4}$$

and

$$h_2 = h_{old} \times \left(\frac{TOL}{2.0 \times T_2} \right)^{\frac{1}{k+2}} \tag{5}$$

where h_{old} is the step size from the previous block. The order which give the maximum step size $h_{\max}$ in Eqn. (4)-(5) will be the order on the next step. Therefore, the approximation of values y can be simultaneously computed using two point or three point block methods on the new step. In the code, to consider raising the order only can be done after having enough points for the higher order method to be used in the next step.

After a successful step, the new step size is given by

$$\begin{aligned} & h_{new} = C \times h_{\max} \\ & \text{if } \left(h_{new} \geq 2 \times h_{old}\right) \text{ then } h_{new} = 2 \times h_{old} \\ & \text{else } h_{new} = h_{old} \end{aligned} \tag{6}$$

where $C = 0.8$ is a safety factor. Whenever the step failure occurs, the step size is

$$h_{new} = 0.5 \times h_{old} \tag{7}$$

and the order remain unchanged. The strategy propose in the code will allow the block methods to varies the step size and subsequently allow to change the order for the next step.

Numerical Treatments of DDEs

Generally, the 2P3S and 3P4S can be implemented to solve (1) as follows.
For 2P3S,

$$y_{n+m} = y_n + h\sum_{j=0}^{5} \beta_j f\left(x_{n+2-j}, y_{n+2-j}, z_{n+2-j}\right),\ m = 1,2$$

and for 3P4S,

$$y_{n+m} = y_n + h\sum_{j=0}^{7} \beta_j f\left(x_{n+3-j}, y_{n+3-j}, z_{n+3-j}\right),\ m = 1,2,3$$

where $z_{n+i-j} = y_{n+i-j}\left(x-\tau\right),\ i = 2,3$.

Now, we described how the calculation of $y\left(\alpha\right)$ where $\alpha = x - \tau(x, y(x))$ is being carried out. The location of α is sought because the calculation of the delay term depends on this location. We should use the interpolation method which has either the same or higher order than the integration method in order to preserve the desired order of accuracy. Here the delay term is approximated using seven points divided difference interpolation if values of y_{n+m} is obtained by 2P3S. Otherwise, nine points divided difference interpolation is applied if values of y_{n+m} is obtained by 3P4S. In divided difference form, the interpolating polynomial can be written as

$$\begin{aligned} P_n(x) &= y[x_0] + (x - x_0)\, y[x_0, x_1] \\ &+ (x - x_0)(x - x_1)\, y[x_0, x_1, x_2] + \dots \\ &+ (x - x_0)\dots(x - x_{n-1})\, y[x_0, x_1, \dots, x_n] \end{aligned} \quad (8)$$

where

$$y[x_0, x_1, \dots, x_n] = \frac{y[x_1, x_2, \dots, x_n] - y[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}.$$

P-AND Q-STABILITY

In the development of the numerical methods, it is of practical importance to study the stability region for those methods. All the stability regions in this paper were obtained using MAPLE. The common test equations are

$$\begin{aligned} y'(x) &= \lambda y(x) + \mu y(x - \tau),\ x \ge x_0, \\ y(x) &= \varphi(x),\ -\tau \le x \le x_0, \end{aligned} \quad (9)$$

and

$$\begin{aligned} y'(x) &= \mu y(x - \tau),\ x \ge x_0, \\ y(x) &= \varphi(x),\ -\tau \le x \le x_0, \end{aligned} \quad (10)$$

where λ and μ are complex numbers. We consider *h* to be a fixed step size such that $x_n = x_0 + nh$ and $mh = \tau$, $m \in I^+$. Let $H_1 = h\lambda$ and $H_2 = h\mu$, we have the following definitions of P- and Q-stability regions which are proposed by Al-Mutib (1984) and adopted by Ishak et al. (2010).

Definition 1:
For a fixed step size *h* and $\lambda, \mu \in R$ in (9), the region R_P in the $H_1 - H_2$ plane is called the P-stability region if for any $(H_1 - H_2) \in R_P$, the numerical solution of (10) vanishes as $x_n \to \infty$.

Definition 2:
For a fixed step size *h* and $\mu \in C$ in (10), the region R_Q in the complex H_2 plane is called the Q-stability region if for any $H_2 \in R_Q$, the numerical solution of (10) vanishes as $x_n \to \infty$.

In this paper, the stability regions are analyzed for each of the method at step the size ratios. For instance, the 2P3S with step size ratio of *r* and *q* can be described in the following matrix form

$$A_2 Y_{N+2} = A_1 Y_{N+1} + h\sum_{i=0}^{2} B_i F_{N+i} \tag{11}$$

where

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, Y_{N+2} = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix}, Y_{N+1} = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}, F_{N+2} = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}, F_{N+1} = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}, F_N = \begin{bmatrix} f_{n-3} \\ f_{n-2} \end{bmatrix}.$$

For B_0, B_1 and B_2, the elements of the matrices are the integration coefficients of 2P3S for various step size ratio of *r*. For example, the elements of B_0, B_1 and B_2 for $(r = 0.5, q = 0.5)$ are as follows where

$$B_2 = \begin{bmatrix} \frac{4669}{12600} & -\frac{129}{12600} \\ \frac{348}{225} & \frac{67}{225} \end{bmatrix}, B_1 = \begin{bmatrix} -\frac{9856}{12600} & \frac{14525}{12600} \\ \frac{288}{225} & -\frac{125}{225} \end{bmatrix}, B_0 = \begin{bmatrix} -\frac{704}{12600} & \frac{4095}{12600} \\ \frac{32}{225} & -\frac{160}{225} \end{bmatrix}.$$

The method in (11) is applied to test equation (9) and (10) respectively in order to obtained the P-stability and Q-stabiliy polynomials. The P-stability polynomial of 2P3S is given by

$$\pi_{2P3S,m}\left(H_1, H_2; t\right) = \det[\left(A_2 - H_1 B_2\right)t^{2+m} - \left(A_1 + H_1 B_1\right)t^{1+m} - H_1 B_0 t^m - H_2 \sum_{i=0}^{2} B_i t^i]$$

and the Q-stability polynomial of 2P3S is given by

$$\varphi_{2P3S,m}\left(H_2; t\right) = \det\left[A_2 t^{2+m} - A_1 t^{1+m} - H_2 \sum_{i=0}^{2} B_i t^i\right].$$

By solving $\pi_{2P3S,1}\left(H_1, H_2; t\right) = 0$ and $\varphi_{2P3S,1}\left(H_2; t\right) = 0$, we obtained the following P-stability and Q-stability regions of 2P3S for various step size ratio of (r, q) in Figure 3 - 5.

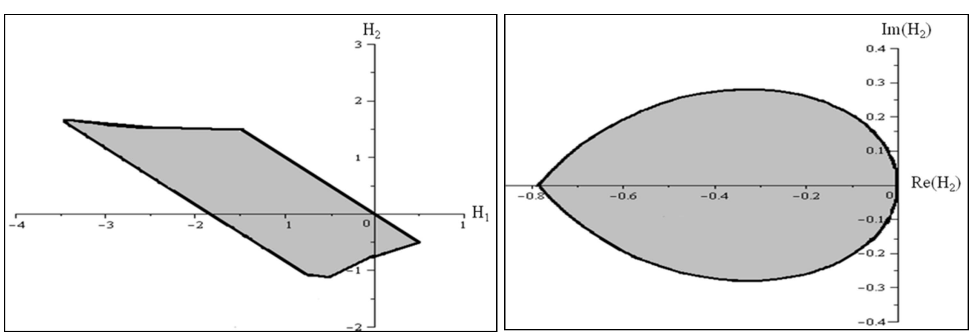

Figure 3: P-stability and Q-stability region for 2P3S when $(r = 1, q = 1)$

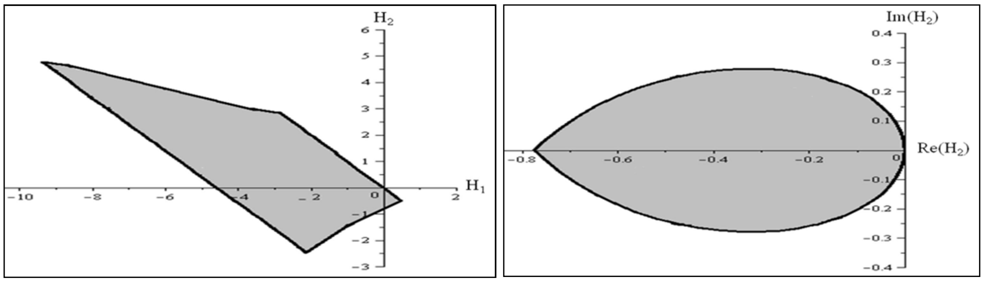

Figure 4: P-stability and Q-stability region for 2P3S when $(r = 2, q = 2)$

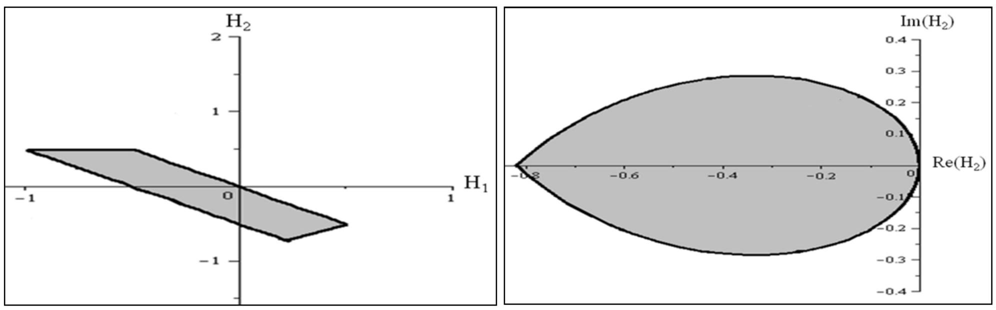

Figure 5: P-stability and Q-stability region for 2P3S when $(r = 0.5, q = 0.5)$

On the other hand, the P-stability and Q-stability regions of 3P4S can be obtained by using the similar technique as illustrated for obtaining the stability regions for 2P3S as above. The following Figure 6 - 8 shows the stability regions of 3P4S.

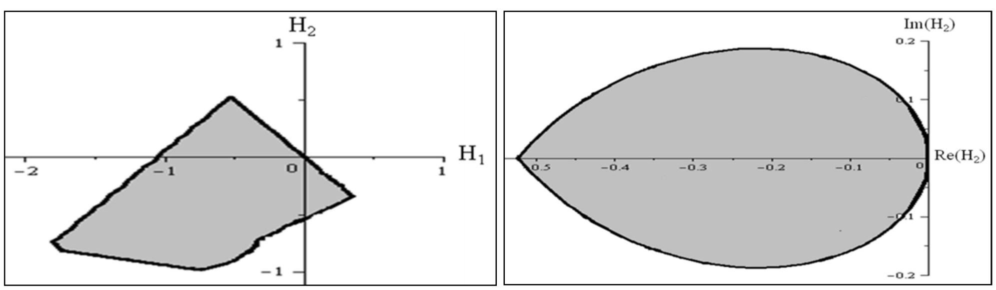

Figure 6: P-stability and Q- stability region for 3P4S when $(r = 1, q = 1, p = 1)$

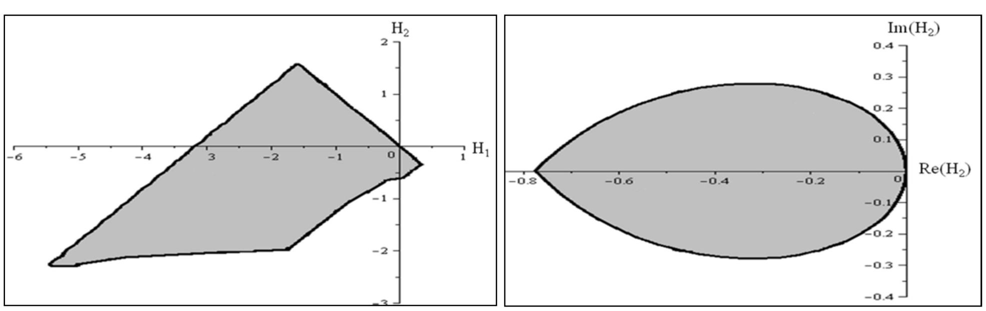

Figure 7: P-stability and Q-stability region for 2P3S when $(r = 2, q = 2, p = 2)$

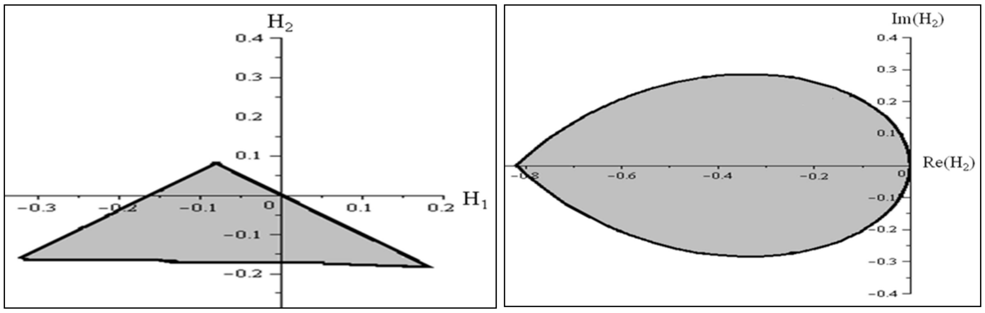

Figure 8: P-stability and Q-stability region for 2P3S when $(r = 0.5, q = 0.5, p = 0.5)$.

The P-stability and Q-stability regions for 2P3S and 3P4S are illustrated in Figure 3-8. The shaded area indicates the stability region for each of the block method. It is clearly showed that the stability region for each of the proposed method is a closed region.

RESULT AND DISCUSSIONS

We test the efficiency of the developed codes at different tolerance (TOL). The code was written in C language.

Problem 1:

$$y'(x) = -y\left(x - \frac{\pi}{2}\right),\ 0 \le x \le 10$$
$$y(x) = \sin(x),\ x \le 0$$

Exact Solution: $y(x) = \sin(x),\ x \ge 0$.

Problem 2:

$$y_1'(x) = -y_1\left(x - \frac{\pi}{2}\right),\ \frac{\pi}{2} \le x \le 10$$
$$y_2'(x) = -y_2\left(x - \frac{\pi}{2}\right),\ \frac{\pi}{2} \le x \le 10$$
$$y_1(x) = \sin(x),\ x \le \frac{\pi}{2},\ y_2(x) = \cos(x),\ x \le \frac{\pi}{2}$$

Exact Solution: $y_1(x) = \sin(x),\ x \ge \frac{\pi}{2},\ y_2(x) = \cos(x),\ x \ge \frac{\pi}{2}$.

The notations used in the tables are as follows:

TS	Total steps taken
FS	Total failure steps

MAXE	Magnitude of the maximum error
AVEERR	Magnitude of the average error
CB(6,8)	Implementation of coupled block method that consists of 2P2S of order six and 3P2S of order eight.
2PBVS	Implementation of two point block method in variable step size technique by Ishak et al. (2010).

Table 1: Comparison between CB(6,8) and 2PBVS for Solving Problem 1

TOL	Method	TS	FS	MAXE	AVERR
10^{-2}	2PBVS	30	0	8.23212(-4)	8.77240(-5)
	CB(6,8)	20	0	2.29552(-4)	1.95274(-5)
10^{-4}	2PBVS	49	2	7.48264(-5)	9.91393(-6)
	CB(6,8)	28	0	3.58248(-7)	5.17767(-8)
10^{-6}	2PBVS	83	3	1.74082(-6)	4.25114(-7)
	CB(6,8)	38	0	2.38519(-9)	5.16375(-10)
10^{-8}	2PBVS	168	5	1.65845(-8)	4.80588(-9)
	CB(6,8)	70	0	8.73552(-10)	1.84701(-10)
10^{-10}	2PBVS	362	5	2.02207(-10)	6.87281(-11)
	CB(6,8)	86	0	8.64521(-10)	2.03400(-10)

Table 2: Comparison between CB(6,8) and 2PBVS for Solving Problem 2

TOL	Method	TS	FS	MAXE	AVERR
10^{-2}	2PBVS	29	0	4.94421(-4)	4.51542(-5)
	CB(6,8)	19	0	1.59708(-5)	1.35850(-6)
10^{-4}	2PBVS	45	0	6.64739(-6)	8.31195(-7)
	CB(6,8)	28	0	2.80670(-7)	5.72088(-8)
10^{-6}	2PBVS	80	0	7.37809(-8)	1.34880(-8)
	CB(6,8)	38	0	2.26106(-9)	3.60828(-10)
10^{-8}	2PBVS	161	0	7.98472(-10)	1.79650(-10)
	CB(6,8)	63	0	7.76393(-10)	1.59609(-10)
10^{-10}	2PBVS	358	0	8.25396(-12)	2.09949(-12)
	CB(6,8)	77	0	7.77750(-10)	1.61642(-10)

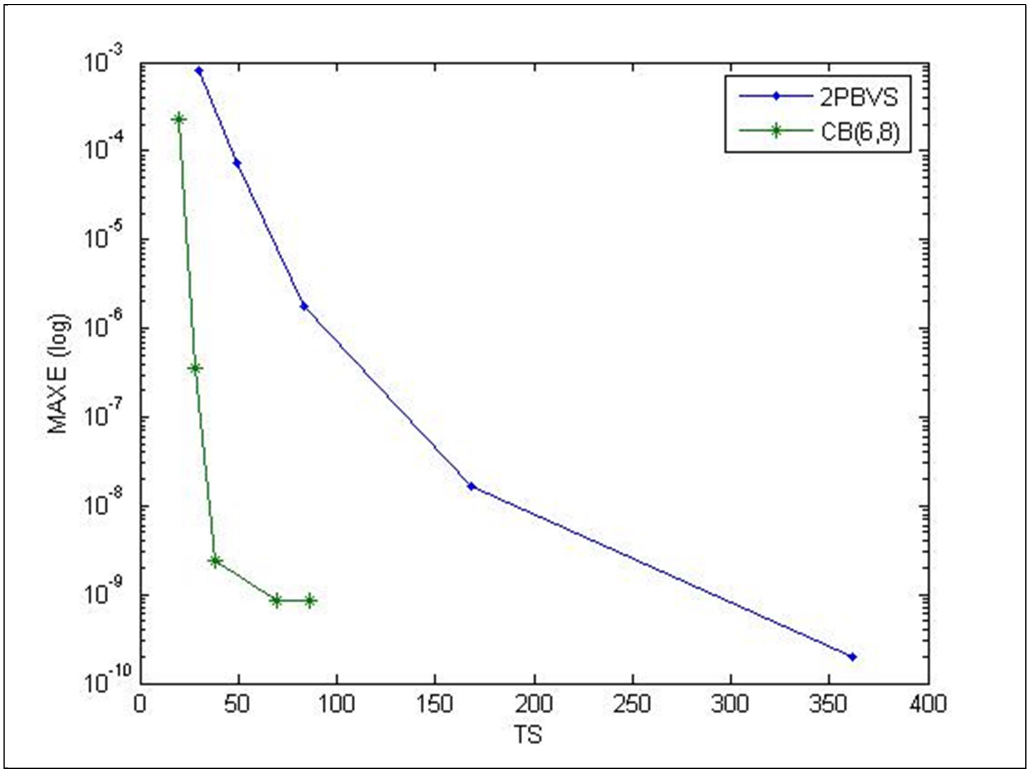

Figure 9: Comparison of maximum error and total step for Problem 1

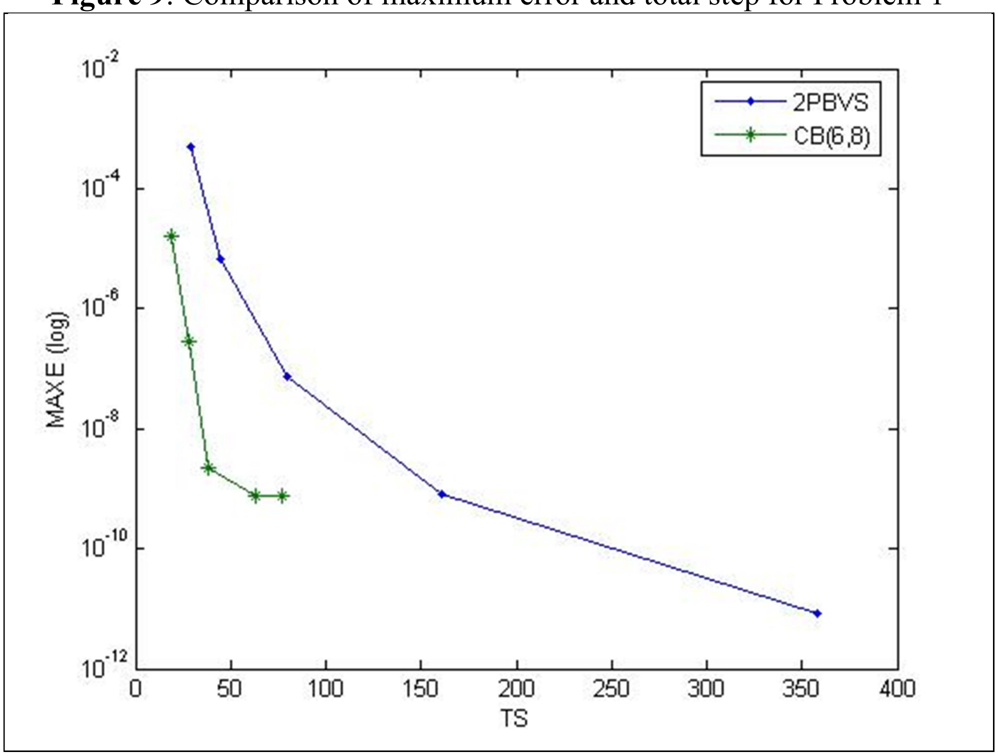

Figure 10: Comparison of maximum error and total step for Problem 2

The numerical results in Table 1-2 clearly showed that the total number of steps taken by CB(6,8) is less than the total number of steps taken by 2PBVS at all tolerances. It is obvious that CB(6,8) has greater reduction in the total number of steps compared to 2PBVS at smaller tolerances in all the given problems. This is expected since the variable step size and order strategy is employed in CB(6,8) code, where the code is allowed to move using two or three points for the next step. We also observed that there is no failure steps in CB(6,8) compared to 2PBVS in Problem 1.

It can be observed that at most of the tolerances, the maximum error and average error of CB(6,8) are superior than 2PBVS for solving the given problems. This is because of CB(6,8) is a combination of block method of order six and eight, while 2PBVS is a block method of order five only. Figure 9-10 display the comparison of the maximum errors versus the total steps for solving Problem 1-2.

CONCLUSION

In this paper, we have presented a variable step size and order code for the numerical solution of DDEs using coupled block method. Hence, we can conclude that the proposed code is efficient and accurate for solving DDEs.

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