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Second Order of Volterra Integro-Differential Equations using Direct Two-Point Hybrid Block Method

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ABSTRACT

A direct method is implemented for the numerical solution of second-order Volterra integro-differential equations (VIDEs). The formulation of two-point hybrid block method will be discussed in this paper to solve second order VIDEs directly without reducing the equations into first order system. The proposed method of order four will calculate the computing solutions using constant step size. The quadrature rule has been used to approximate the integral part. Numerical results of linear and nonlinear VIDEs are presented and show that the hybrid block method is appropriate for solving second order VIDEs directly.

Keywords: Direct method, Hybrid block method, Volterra integro-differential equations.

INTRODUCTION

Integro-differential equations emerge in many applications in natural sciences and engineering. Several techniques including Implicit runge-kutta Nystrom method (Brunner, H., 1987), one-step increment method (Garey & Shaw, 1991), Homotopy analysis method (Atabakan et al., 2012), reproducing kernel Hilbert space (Altawallbeh et al., 2013) and Haar wavelets (Aziz et al., 2015) have been applied to investigate second order VIDEs directly. Note that solving second order VIDEs also can be reduced into system of integro-differential equations of the first order.

Recently, (Mehdiyeva et al., 2015) proposed a second derivative method with special techniques adapted for solution of second order VIDEs. The motivation of this research is to proposed direct hybrid two-point one-step block method for solving second order VIDEs directly using constant step size. Previously, most of existing methods will compute only one solution at one step, while the proposed method will compute two-point solutions simultaneously in a block.

Hence, the direct hybrid one-step block method with one off-step point of order four will be implemented to solve differential part of VIDEs directly while the integral part is solved using numerical quadrature rule. The direct hybrid one-step block method has the advantages of reducing total number of steps and function evaluations but still manage to establish better or comparable accuracy when compared to the existing method. Consider the general second order Volterra integro-differential equation as follow:

$$y''(x) = f(x, y(x), y'(x), z(x)), \;\; y(0) = b_0, y'(0) = b_1$$

where

$$z(x) = \int_0^x K\big(x, y(s), y'(s)\big)ds, \quad 0 \le x \le a. \tag{1}$$

FORMULATION OF THE METHOD

Derivation of the Method

In this section, the mathematical formulation of the hybrid one-step block method that based on numerical integration has been discussed. The derivation involves both which is main method and one off-step point method.

In the main method, two approximations values of y_{n+1} and y_{n+2} will computed simultaneously. The first point y_{n+1} will be approximated by integrating once and twice over the interval $[x_n, x_{n+1}]$,

$$\int_{x_n}^{x_{n+1}} y''(x)dx = \int_{x_n}^{x_{n+1}} f(x, y, y')dx, \tag{2}$$

$$\int_{x_n}^{x_{n+1}} \int_{x_n}^{x} y''(x)dxdx = \int_{x_n}^{x_{n+1}} \int_{x_n}^{x} f(x, y, y')dx. \tag{3}$$

The same process will be implemented to obtain the second point y_{n+2} of the hybrid one-step block method. The equations will be integrated once and twice over the interval

$[x_{n+1}, x_{n+2}]$,

$$\int_{x_{n+1}}^{x_{n+2}} y''(x)dx = \int_{x_{n+1}}^{x_{n+2}} f(x, y, y')dx, \tag{4}$$

$$\int_{x_{n+1}}^{x_{n+2}} \int_{x_{n+1}}^{x} y''(x)dxdx = \int_{x_{n+1}}^{x_{n+2}} \int_{x_{n+1}}^{x} f(x, y, y')dx. \tag{5}$$

Next, the Lagrange polynomial technique will be applied to interpolate the function $f(x, y, y')$ in (2) to (5). The interpolating point involved is $\{x_n, x_{n+\frac{1}{2}}, x_{n+1}, x_{n+2}\}$. Then, by taking $s = \frac{x - x_{n+2}}{h}$ and replace $dx = hds$ and now changing the limit of integration from -2 to -1 for the first point and -1 to 0 for the second point. Hence, the corrector formulae for direct hybrid one-step block method of order four is obtained as follows:

$$y'_{n+1} = y'_n + \frac{h}{6}\left(f_n + 4f_{n+\frac{1}{2}} + f_{n+1}\right)$$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{360}\left(57f_n + 128f_{n+\frac{1}{2}} - 6f_{n+1} + f_{n+2}\right)$$

$$y'_{n+2} = y'_{n+1} + \frac{h}{6}\left(f_n - 4f_{n+\frac{1}{2}} + 7f_{n+1} + 2f_{n+2}\right)$$
$$y_{n+2} = y_{n+1} + hy'_{n+1} + \frac{h^2}{360}\left(27f_n - 112f_{n+\frac{1}{2}} - 234f_{n+1} + 31f_{n+2}\right)$$

(6)

The same process was applied to derive the formula for off-step point method of order three. In this process, the interpolation point involved is $\{x_n, x_{n+1}, x_{n+2}\}$. Integrating $f(x, y, y')$ from x_n to x_{n+2} and replace it with interpolation polynomial technique and hence, the predictor formula of the off-step point method is obtained as follows:

$$y'_{n+\frac{1}{2}} = y'_n + \frac{h}{8}(3f_n + f_{n+1})$$
$$y_{n+\frac{1}{2}} = y_n + hy'_n + \frac{h^2}{8}(-f_n + 3f_{n+1} + 8f_{n+2})$$

(7)

The proposed method in (6) is derived as a main method and (7) as an additional method which are combined and used as hybrid block one-step method.

Order of the Method

The formula for the constants C_q will be applied to determine the order of this method. the general formula is defined as follows:

$$C_0 = \sum_{j=0}^{k} \alpha_j,$$
$$C_1 = \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j - \sum_{j=1}^{k} \beta v_j\,,$$
$$\vdots$$
$$C_q = \frac{1}{q!}[\sum_{j=0}^{k} j^q \alpha_j - q(\sum_{j=0}^{k} j^{q-1}\beta_j + \sum_{j=1}^{k} vj^{q-1}\beta_{vj})]\,,$$

(8)

where q=2, 3, 4, …The order of the method in (6) is determined by applying the formula in (8).

For $q = 0$

$$C_0 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For $q = 1$

$$C_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For $q = 2$:

$$C_2 = \frac{1}{2!}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$
$$+ \begin{bmatrix} 0 \\ \frac{1}{360} \\ 0 \\ \frac{31}{360} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{6}{360} \\ 0 \\ \frac{234}{360} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{128}{360} \\ 0 \\ -\frac{112}{360} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{57}{360} \\ 0 \\ \frac{27}{360} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For $q = 3$:
$C_3 = [0 \quad 0 \quad 0 \quad 0]^T.$

For $q = 4$:
$C_4 = [0 \quad 0 \quad 0 \quad 0]^T.$

For $q = 5$:
$C_5 = [0 \quad 0 \quad 0 \quad 0]^T.$

For $q = 6$:
$C_6 = \left[0 \quad \frac{21}{1880} \quad 0 \quad \frac{3}{1880}\right]^T \neq [0 \quad 0 \quad 0 \quad 0]^T.$

The method has order p if $C_0 = C_1 = \cdots = C_{p+1} = 0$ and $C_{p+2} \neq 0$. The value for C_{p+2} is the error constant and the corrector formulae direct hybrid one-step block method in equation (6) is of order four. A method is said to be consistent if it has order at least one.

Stability Analysis

The method is zero stable provided the roots R_j of the first characteristic polynomial where:

$$\rho(R) = \det\left[\sum_{j=0}^{k} A^i R^{(k-i)}\right] = 0$$

satisfy with $|R_j| \leq 1$ and those roots with $|R_j| = 1$.

The first characteristic polynomial of the method is given as follows:

$$\rho(R) = det[RA^0 - A^1] = 0$$

$$A^0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(R) = \ det \begin{bmatrix} R & 0 & 1 & 0 \\ 0 & R & 0 & 1 \\ R & 0 & R & 0 \\ 0 & R & 0 & R \end{bmatrix} = \ 0\ , R^4 = 0{,}0{,}0{,}0.$$

Since $|R_j| \leq 1$, the method is said to be zero stable.

IMPLEMENTATION

Since in VIDEs both differential and integral operator appeared together in the same equation. The implementation of the method will be based on the combination of direct hybrid one-step block method (D2HBM) with quadrature rule. The Euler method and off-step point method will be used as the initial starting point for each block as a predictor.

In this paper, for solving (1) the combination of D2HBM with modified composite Simpson's 1/3 rule has been applied. The application of hybrid one-step method will be based on predictor and corrector formula. The D2HBM is applied to the differential part of VIDE as follows:

$$y'_{n+1} = y'_n + \frac{h}{6}\left(f(x_n, y_n, y'_n, z_n) + 4f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}, y'_{n+\frac{1}{2}}, z_{n+\frac{1}{2}}) + f(x_{n+1}, y_{n+1}, y'_{n+1}, z_{n+1}\right),$$

$$\begin{aligned} y_{n+1} = y_n + hy'_n &+ \frac{h^2}{360}\Big(57f(x_n, y_n, y'_n, z_n) + 128f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}, y'_{n+\frac{1}{2}}, z_{n+\frac{1}{2}}) \\ &- 6f(x_{n+1}, y_{n+1}, y'_{n+1}, z_{n+1}) + f(x_{n+2}, y_{n+2}, y'_{n+2}, z_{n+2})\Big), \end{aligned}$$

$$\begin{aligned} y'_{n+2} = y'_{n+1} &+ \frac{h}{6}\Big(f(x_n, y_n, y'_n, z_n) - 4f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}, y'_{n+\frac{1}{2}}, z_{n+\frac{1}{2}}) \\ &+ 7f(x_{n+1}, y_{n+1}, y'_{n+1}, z_{n+1}) + 2f(x_{n+2}, y_{n+2}, y'_{n+2}, z_{n+2})\Big), \end{aligned}$$

$$\begin{aligned} y_{n+2} = y_{n+1} + hy'_{n+1} &+ \frac{h^2}{360}\Big(27f(x_n, y_n, y'_n, z_n) - 112f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}, y'_{n+\frac{1}{2}}, z_{n+\frac{1}{2}}) \\ &+ 234f(x_{n+1}, y_{n+1}, y'_{n+1}, z_{n+1}) + 31f(x_{n+2}, y_{n+2}, y'_{n+2}, z_{n+2})\Big). \end{aligned} \tag{9}$$

A suitable numerical quadrature rules are used to calculate the integral part of VIDEs. The values for z_{n+1} and z_{n+2} are calculated by applied modified composite Simpson's rule with interpolation schemes. Given $n = 0,2,4,6,\ldots$, we can formulate,

$$
\begin{aligned}
z_{n+1} &= \frac{h}{3}\sum_{i=0}^{n} \omega_s^i K(x_{n+1}, x_i, y_i, y_i') \\
&\qquad + \frac{h}{6}\left[K(x_{n+1}, x_n, y_n, y_n') + 4K\left(x_{n+1}, x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}, y'_{n+\frac{1}{2}}\right)\right. \\
&\qquad \left. + K(x_{n+1}, x_{n+1}, y_{n+1}, y'_{n+1})\right],
\end{aligned}
$$

$$
z_{n+2} = \frac{h}{3}\sum_{i=0}^{n+2} \omega_s^i K(x_{n+2}, x_i, y_i, y_i'). \tag{10}
$$

where ω_s^i are the Simpson's rule weights 1,4,2,...,4,1. Values of $z_{n+\frac{1}{2}}$ is calculated by using modified trapezoidal rule,

$$
z_{n+\frac{1}{2}} = z_n + \frac{h}{4}\left[K\left(x_{n+\frac{1}{2}}, x_n, y_n, y_n'\right) + K\left(x_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}, y'_{n+\frac{1}{2}}\right)\right]. \tag{11}
$$

NUMERICAL RESULTS

Four numerical examples of second-order VIDEs are used to demonstrate the method. All the numerical results have been solved using the hybrid block method and programs were written in C language.

The notations used in the following tables

h	Step size used
MAXE	Maximum error
TFC	Total function calls
TS	Total steps
D2HBM	Direct two-point hybrid block method with Simpson's rule proposed in this paper
Method A	One-step method and one-step increment method with trapezoidal rule by (Garey & Shaw, 1991)
Method B	One-step method and one-step increment method with composite trapezoidal rule by (Garey & Shaw, 1991)

Problem 1: The nonlinear of VIDE (Aziz et al., 2015)

$$y''(x) = \frac{\sinh(x)\,1}{2} coshxsinhx - \frac{1}{2}x - \int_0^x y^2(t)dt$$

subject to IVP condition: $y(0) = 0, y'(0) = 1, \quad 0 \le x \le 1.$

Exact solution: $(x) = sinh\, x.$

Table 1: Comparison of the numerical results for solving Problem 1

h	**METHOD**	**MAXE**	**TFC**	**TS**	**TIME**
0.1	Method A	1.6660E-01	80	10	0.4103
	Method B	9.2250E-04	80	10	0.4101
	D2HBM	3.6666E-05	70	5	0.3019
0.01	Method A	3.4567E-02	800	100	0.7948
	Method B	6.5497E-04	800	100	0.8014
	D2HBM	4.6649E-06	760	50	0.5947
0.001	Method A	7.3355E-02	8000	1000	4.1494
	Method B	4.6880E-05	8000	1000	4.2091
	D2HBM	7.6549E-07	7560	500	3.8919

Problem 2: The linear of VIDE (Taylor & Tarang, 2010).

$$y''(x) = y(x) + y'(x) + \int_0^x -sinx - cosx - e^x + y(t)dt$$

subject to IVP condition: $y(0) = 1, y'(0) = 1, \quad 0 \le x \le 1.$

Exact solution: $y(x) = \frac{sinx+cosx+e^x}{2}$

Table 2: Comparison of the numerical results for solving Problem 2

h	**METHOD**	**MAXE**	**TFC**	**TS**	**TIME**
0.1	Method A	4.6649E-02	80	10	0.3212
	Method B	6.5597E-05	80	10	0.3112
	D2HBM	2.1154E-06	70	5	0.2802
0.01	Method A	9.5579E-02	800	100	0.6898
	Method B	5.5549E-06	800	100	0.6977
	D2HBM	7.5249E-06	760	50	0.4791
0.001	Method A	2.1166E-03	8000	1000	4.9149
	Method B	6.1203E-06	8000	1000	4.9418
	D2HBM	3.4457E-08	7560	500	4.6144

Problem 3: The linear of VIDE (Garey & Shaw, 1991)

$$y''(x) = y(x)(4x^2 + 2) - x\left(1 - e^{\frac{x^2}{2}}\right) - \int_0^x xt\left(y'(t) + y(t)(1+2t)^{\frac{1}{2}}\right)dt$$

subject to IVP condition: $y(0) = 1, y'(0) = 0, \quad 0 \le x \le 1.$

Exact solution: $y(x) = e^{x^2}$

Table 3: Comparison of the numerical results for solving Problem 3

h	**METHOD**	**MAXE**	**TFC**	**TS**	**TIME**
0.1	Method A	1.7192E-02	80	10	0.3914
	Method B	3.9191E-04	80	10	0.4001
	D2HBM	7.2251E-05	70	5	0.3011
0.01	Method A	1.2649E-02	800	100	0.7014
	Method B	4.6659E-04	800	100	0.7119
	D2HBM	6.5559E-06	760	50	0.4559
0.001	Method A	7.6659E-03	8000	1000	4.3597
	Method B	7.6998E-05	8000	1000	4.2118
	D2HBM	1.2254E-08	7560	500	3.6584

Problem 4: The linear of VIDE (Brunner, 1987)

$$y''(x) = -\frac{1}{1+x}y(x) + \sin(x)\,y'(x) + \int_0^x -\frac{1+t}{1-t^2}y(t) + y'(t)dt$$

subject to IVP condition: $y(0) = 1, y'(0) = 0, \quad 0 \le x \le 1.$

Exact solution: $y(x) = \frac{x}{1+x}$.

Table 4: Comparison of the numerical results for solving Problem 4

$\boldsymbol{h}$	**METHOD**	**MAXE**	**TFC**	**TS**	**TIME**
0.1	Method A	9.3359E-01	80	10	0.3155
	Method B	8.0112E-04	80	10	0.3248
	D2HBM	6.3359E-05	70	5	0.2658
0.01	Method A	4.6659E-02	800	100	0.7598
	Method B	8.5549E-04	800	100	0.8144
	D2HBM	3.4457E-06	760	50	0.4197
0.001	Method A	7.2449E-03	8000	1000	4.5878
	Method B	4.1560E-05	8000	1000	4.6678
	D2HBM	7.6659E-08	7560	500	3.3315

There are four second order VIDEs problems have been solved using D2HBM method in order to validate the accuracy and efficiency of the method. Tables 1-4 shown the comparison results between D2HBM, Method A and Method B. All the numerical results used three different step sizes which is $h = 0.1, 0.01, 0.001$. Based on the results, the D2HBM and Method B is better compared to Method A in term of maximum error.

Then, the D2HBM method is slightly better compared to Method A and Method B in term of total function calls and number of total steps. From the execution time can be concluded that the D2HBM is less expensive than Method A and Method B. It is apparent that the D2HBM manages to achieve better accuracy as the step size getting smaller. Prove that the D2HBM is suitable for solving second order VIDEs directly compared with the existing methods.

CONCLUSION

The main idea of this paper is to solve second order VIDEs without reducing the equations into first order system. The proposed method is appropriate for solving linear and nonlinear of second order VIDEs directly. Thus, combination of D2HBM method with composite Simpson's rule is provide a better maximum error. The accuracy of the D2HBM for solving the tested problems is improved as the step sizes reduced.

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