Menemui Matematik (Discovering Mathematics)
Vol. 40, No. 2: 89 - 101 (2018)

# Isomorphism theorems in Group Theory

Mohamad Bakri Zubir
*Institut Sains Matematik, Universiti Malaya 50603 Kuala Lumpur*
mbakri@um.edu.my

## ABSTRACT

The purpose of this write-up to provide a leisurely introduction to homomophisms and isomorphims in Group Theory. Instead of simply stating the theorems, we discuss how isomorphism theorems are formulated from homomorphisms. Relevant examples are presented to aid understanding of the theorems. Only rudimentary knowledge in functions, number theory and modern algebra is needed, a bit of mathematical sophistication will be more than adequate.

**Keywords: groups, homomorphism, isomorphisms**

## PRELIMINARIES

**Definition 1**. A group is a pair $(G,*)$ consisting of a nonempty set $G$ and a binary operation $*$ defined on $G$, satisfying the four requirements:

(1) $G$ is closed under the operation $*$,
(2) the operation $*$ is associative,
(3) $G$ contains an identity element $e$ for the operation $*$, and
(4) each element $a$ of $G$ has an inverse $a^{-1} \in G$, relative to $*$.

**Definition 2**. Let $(G,*)$ be a group and $H \subseteq G$ be a nonempty subset of $G$. The pair $(H,*)$ is said to be a subgroup of $(G,*)$ if $(H,*)$ is itself a group.

**Definition 3**. Let $(H,*)$ be a subgroup of the group $(G,*)$ and let $a \in G$. The set

$$a * H = \{ a * h \mid h \in H\}$$

is called a left coset of $H$ in $G$. The element $a$ is a representative of $a * H$. The right cosets $H * a$ of $H$ are defined similarly.

**Definition 4**. A subgroup $H$ of $(G,*)$ is called a normal in $G$ if it closed with respect to conjugates; that is, if

$$\text{for any } a \in H \text{ and } x \in G,\ x * a * x^{-1} \in H.$$

If $H$ is normal in $G$ then $H * a = a * H$.

The collection of distinct cosets of $H$ in $(G,*)$, denoted $G/H$ is

$$G/H = \{H * a \mid a \in G\}.$$

A rule of composition $\otimes$ may be defined on $G/H$ by the formula

$$(H * a) \otimes (H * b) = H * (a * b)$$

so that $(G/H,\otimes)$ is a group.

## HOMOMORPHISM

We start with a homomorphism between two groups.

Let $(G,*)$ and $(G_1,\circ)$ be two groups and $f$ a function from $G$ into $G_1$. (The function $f$ is neither assumed to be onto nor one-to-one.) Then $f$ is said to be a homomorphism (or operation-preserving function) from $(G,*)$ into $(G_1,\circ)$ if and only if

$$f(a*b) = f(a)\circ f(b)$$

for every pair of elements $a, b \in G$.

Some results concerning homomorphic mappings:

**Theorem 1**. *If $f$ is a homomorphism from the group $(G,*)$ into the group $(G_1,\circ)$, then*

(1) *$f$ maps the identity element $e$ of $(G,*)$ onto the identity element $e_1$ of $(G_1,\circ)$, that is $f(e) = e_1$,*
(2) *$f$ maps the inverse of an element $a \in G$ onto the inverse of $f(a)$ in $(G_1,\circ)$, that is $f(a^{-1}) = f(a)^{-1}$ for each $a \in G$.*

**Theorem 2**. *Let $f$ be a homomorphism from the group $(G,*)$ into the group $(G_1,\circ)$. Then*

(1) *for each subgroup $(H,*)$ of $(G,*)$, the pair $(f(H),\circ)$ is a subgroup of $(G_1,\circ)$,*

(2) *for each subgroup $(K,\circ)$ of $(G_1,\circ)$, the pair $(f^{-1}(K),*)$ is a subgroup of $(G,*)$.*

**Corollary 1**.

(1) *If $f(G) = G_1$, then for each normal subgroup $(H,*)$ of $(G,*)$, the subgroup $(f(H),\circ)$ is normal in $(G_1,\circ)$.*

(2) *For each normal subgroup $(K,\circ)$ of $(G_1,\circ)$, the subgroup $(f^{-1}(K),*)$ is normal in $(G,*)$.*

**Definition 5**. Let $f$ be a homomorphism from the group $(G,*)$ into the group $(G_1,\circ)$ and let $e_1$ be the identity element of $(G_1,\circ)$. The kernel of $f$, denoted by $\ker(f)$, is the set

$$\ker(f) = \{ a \in G \mid f(a) = e_1\}.$$

**Theorem 3**. *Let $f$ be a homomorphism from the group $(G,*)$ into the group $(G_1,\circ)$. Then $f$ is one-to-one if and only if $\ker(f) = \{e\}$.*

**Theorem 4**. *If $f$ is a homomorphism from the group $(G,*)$ into the group $(G_1,\circ)$, then the pair $(\ker(f),*)$ is a normal subgroup of $(G,*)$.*

By Theorem 4 every homomorphism determines a normal subgroup by means of its kernel. On the other hand, every normal subgroup give rise to a homomorphic mapping, the so-called natural (canonical or natural) mapping. The problems of finding homomorphisms and normal subgroups are inseparable.

**Theorem 5**. *Let $(H,*)$ be a normal subgroup of the group $(G,*)$. Then the mapping $\mathrm{nat}_H: G \to G/H$ defined by*

$$\mathrm{nat}_H(a) = H * a$$

*is a homomorphism from $(G,*)$ onto the quotient group $(G/H,\otimes)$ where the kernel of $\mathrm{nat}_H$ is precisely the set $H$.*

Theorem (5) is rephrased so that no reference is made to the notion of quotient group:

**Theorem 6**. *Let $(H,*)$ be a normal subgroup of the group $(G,*)$. Then there exists a group $(G_1,\circ)$, and a homomorphism $f$ from $(G,*)$ onto $(G_1,\circ)$ such that* $\ker(\mathrm{f}) = H$.

Here we take $(G_1,\circ)$ to be the quotient group $(G/H,\otimes)$ and $f = \mathrm{nat}_H$.

## ISOMORPHISMS

**Definition 6**. Two groups $(G,*)$ and $(G_1,\circ)$ are said to be isomorphic, denoted $(G,*) \simeq (G_1,\circ)$, if there exists a one-to-one homomorphism $f$ of $(G,*)$ onto $(G_1,\circ)$, that is $f(G) = G_1$. Such a homomorphism is called an isomorphism, or isomorphic mapping, of $(G,*)$ onto $(G_1,\circ)$.

## THE FUNDAMENTAL THEOREM OF HOMOMORPHISMS

Let $f$ be a homomorphism of a group $G$ onto a group $G_1$, $H$ be a normal subgroup of $G$. By Theorem 5, there is a natural homomorphism $\mathrm{nat}_H: G \to G/H$ defined by $\mathrm{nat}_H(a) = H * a$. We would like to define a homomorphism $h$ of $G/H$ into $G_1$.(Figure 1). For any $g \in G_1$ there is $a \in G$ such that $g_1 = f(a)$. With such $a$, we have

$H * a \in G/H$. We may define a homomorphism $h$ of $G/H$ onto $G_1$ by

$$h(H * a) = f(a),$$

and since $\mathrm{nat}_H(a) = H * a$, we have

$$(h \circ \mathrm{nat}_H)(a) = f(a).$$

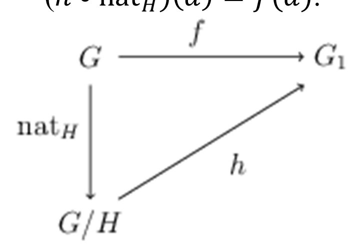


FIGURE 1

What conditions must we have so that the homomorphism $h$ is defined? For a start the function $h$ must be well defined, that is if $H * a = H * b$ then $f(a) = f(b)$. Suppose $H * a = H * b$. Then $ab^{-1} \in H$. Now if $H \subseteq \ker(f)$ we would have $a * b^{-1} \in \ker(f)$ or $f(a) = f(b)$, so the mapping would be well defined.

Since $H$ is a normal subgroup of $G$, the mapping $h$ is a homomorphism because

$$h(Ha \otimes Hb) = h(H * a * b) = f(a * b) = f(a) \circ f(b) = h(H * a) \circ (H * b).$$

The homomorphism $h$ is clearly onto because given any $g \in G_1$, since $f$ is onto, there is an $a \in G$ such that $g = f(a)$ and so $f(a) = h(H * a)$.

How about uniqueness? Suppose there is some homomorphism $h'$ from $G/H$ onto $G_1$ such that $f = h' \circ \mathrm{nat}_H$. Then

$$h(H * a) = f(a) = (h' \circ \mathrm{nat}_H)(a) = h'(\mathrm{nat}_H(a)) = h'(H * a),$$

so that such a function $h$ is unique.

Collecting the results we have our first theorem.

**Theorem 7**. *Let $f$ be a homomorphism of a group $G$ onto a group $G_1$, $H$ be a normal subgroup of $G$ such that $H \subseteq \ker(f)$, and $\mathrm{nat}_H$ be the natural homomorphism of $G$ onto $G/H$. Then there exists a unique homomorphism $h$ of $G/H$ onto $G_1$ such that $f = h \circ \mathrm{nat}_H$. Furthermore, $h$ is one-one if and only if $H = \ker(f)$.*

We prove $h$ is one to one if and only if $H = \ker(f)$. Suppose $h$ is one-one. We show $\ker(f) \subseteq H$. Suppose $a \in \ker(f)$. Then $Ha = f(a) = e_1 = f(e) = He = H$. This implies $a \in H$ and since $H \subseteq \ker(f)$, we have $H = \ker(f)$.

Conversely suppose $H = \ker(f)$. We show $h$ is one to one. Suppose $f(a) = f(b)$. This implies $ab^{-1} \in \ker(f) = H$ so that $Ha = Hb$.

□

It follows that if $f$ is an onto homomorphism and $H = \ker(f)$, $h$ is an isomorphism and hence $G/\ker(f)$ is isomorphic to $G_1$. This result is known as the **fundamental theorem of homomorphisms** for groups.

**Example 1**. Let $f$ be the homomorphism of $(\mathbb{Z}, +)$ onto $(\mathbb{Z}_3, +_3)$ defined by $f(n) = [n]$ for all $n \in \mathbb{Z}$, where $[n]$ is the set of all integers which have the same remainder upon division by 3 (Figure 2).

Let $\mathrm{nat}_{\mathbb{Z}/\langle 6\rangle}$ be the natural homomorphism of $\mathbb{Z}$ onto $\mathbb{Z}/\langle 6\rangle$. Since $\mathbb{Z}$ is an abelian group, the subgroup $\langle 6\rangle$ is a normal subgroup of $\mathbb{Z}$. The kernel of $f$ is $\ker(f) = \langle 3\rangle$, and
$\langle 6\rangle \subseteq \ker(f)$. Thus there is a homomorphism $h$ of $\mathbb{Z}/\langle 6\rangle$ onto $\mathbb{Z}_3$ such that

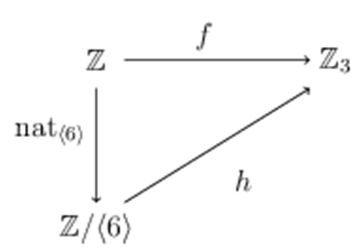


FIGURE 2

$f = h \circ \mathrm{nat}_{\mathbb{Z}/\langle 6\rangle}$. The homomorphism is defined by $h(n + \langle 6\rangle) = [n]$.

Here

$$\begin{aligned}\langle 3\rangle &= \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}\\ \langle 6\rangle &= \{\dots, -18, -12, -6, 0, 6, 12, 18, \dots\}\\ \mathbb{Z}/\langle 6\rangle &= \{0 + \langle 6\rangle, 1 + \langle 6\rangle, 2 + \langle 6\rangle, 3 + \langle 6\rangle, 4 + \langle 6\rangle, 5 + \langle 6\rangle\}\\ \mathbb{Z}_3 &= \{[0], [1], [2]\},\end{aligned}$$

and the homomorphism is

$$\begin{aligned}&h(0 + \langle 6\rangle) = [0], && h(1 + \langle 6\rangle) = [1], && h(2 + \langle 6\rangle) = [2],\\ &h(3 + \langle 6\rangle) = [3] = [0], && h(4 + \langle 6\rangle) = [4] = [1], && h(5 + \langle 6\rangle) = [5] = [2].\end{aligned}$$

## THE FIRST ISOMORPHISM THEOREM

For the general case, let $f$ be a homomorphism of a group $G$ into a group $G_1$. We do not assume $f$ to be onto. However $f: G \to f(G)$ is onto, and by the fundamental theorem we obtain (Figure 3)

**Theorem 8** (First Isomorphism Theorem). *Let $f$ be a homomorphism of a group $G$ into a group $G_1$. Then $f(G)$ is a subgroup of $G_1$ and*

$$G/\ker(f) \simeq f(G).$$

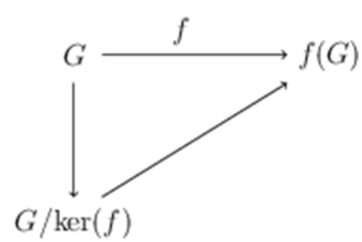


FIGURE 3

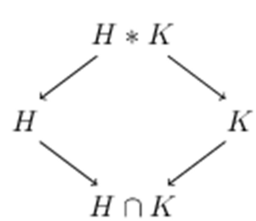


FIGURE 4

## THE SECOND ISOMORPHISM THEOREM

We may also consider homomorphism between subgroups of a group $(G,*)$.

Let $H$ and $K$ be subgroups of a group $G$ (Figure 4). The set $H * K = \{h * k \mid h \in H, k \in K\}$ is not a subgroup of $G$. If we impose the condition that $K$ is normal then $H * K$ is a subgroup. This is because for any $h_1 * k_1$ and $h_2 * k_2$ in $H * K$,

$$\begin{aligned}(h_1 * k_1) * (h_2 * k_2)^{-1} &= h_1 * k_1 * k_2^{-1} * h_2^{-1} \\ &= \underbrace{(h_1 * h_2^{-1})}_{\in H} * \underbrace{(h_2 * k_1 * h_2^{-1})}_{\in K} * \underbrace{(h_2 * k_2^{-1} * h_2^{-1})}_{\in K} \\ &\in HK.\end{aligned}$$

Now $K \subseteq H * K$ because for any $h \in K, k = e * k \in H * K$. It is also a normal subgroup of $H * K$ because for any $k \in K$ and $h_1 * k_1 \in H * K$,

$$\begin{aligned}(h_1 * k_1) * k * (h_1 * k_1)^{-1} &= (h_1 * k_1) * k * (k_1^{-1} * h_1^{-1}) \\ &= \underbrace{(h_1 * k_1 * h_1^{-1})}_{\in K} * \underbrace{h_1 * (k * k_1^{-1}) * h_1^{-1})}_{\in K} \\ &\in K.\end{aligned}$$

We can therefore form the quotient group $H * K/K$ where an element is of the form $(h * k) * K = h * K$.

We may then define a homomorphism $f$ from $H$ to $H * K/K$ by $f(h) = h * K$ (Figure 5). It is easily checked that it is well defined, onto homomorphism.

By Theorem (8), $H/\ker(f) \simeq H * K/K$ . Now

$$\begin{aligned}\ker(f) &= \{h \in H \mid h * K = \text{ identity element of } H * K/K\} \\ &= \{h \in H \mid h * K = K\} \\ &= \{h \in H \mid h \in K\} \\ &= H \cap K.\end{aligned}$$

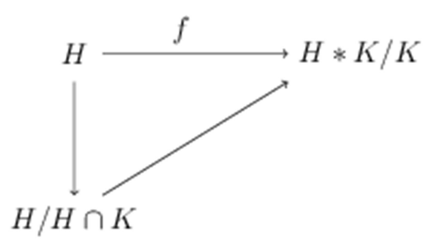


**FIGURE 5**

Thus we obtain

**Theorem 9** (Second Isomorphism Theorem). *Let $H$ and $K$ be subgroups of a group $G$ with $K$ normal in $G$. Then*

$$H/(H \cap K) \simeq H * K/K.$$

**Example 2**. Consider the group $(\mathbb{Z}, +)$ and its subgroups $H = \langle 2 \rangle$ and $K = \langle 3 \rangle$ (Figure 6). Then $H + K = \langle 2 \rangle + \langle 3 \rangle = \mathbb{Z}$ and $H \cap K = \langle 6 \rangle$. Theorem 9 says that

$$H/(H \cap K) \simeq (H + K)/K.$$

that is

$$\langle 2 \rangle / \langle 6 \rangle \simeq \mathbb{Z}/\langle 3 \rangle.$$

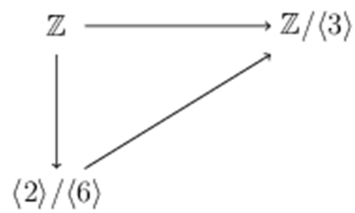


**FIGURE 6**

Here

$$\langle 2\rangle = \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$$
$$\langle 3\rangle = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$$
$$\langle 2\rangle + \langle 3\rangle = \langle 1\rangle = \mathbb{Z}$$
$$\langle 2\rangle \cap \langle 3\rangle = \langle 6\rangle = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$$
$$\langle 2\rangle/\langle 6\rangle = \{0 + \langle 6\rangle, 2 + \langle 6\rangle, 4 + \langle 6\rangle\}$$

and

$$\mathbb{Z}/\langle 3\rangle = \{0 + \langle 3\rangle, 1 + \langle 3\rangle, 2 + \langle 3\rangle\}$$

where the correspondence is

$$0 + \langle 6\rangle \leftrightarrow 0 + \langle 3\rangle$$
$$2 + \langle 6\rangle \leftrightarrow 1 + \langle 3\rangle$$
$$4 + \langle 6\rangle \leftrightarrow 2 + \langle 3\rangle.$$

## THE THIRD ISOMORPHISM THEOREM

Suppose now we have a homomorphism $f$ of a group $G$ *onto* a group $G_1$. Let $H$ and $f(H)$ be the normal subgroup of $G$ and $G_1$ respectively. Let $\text{nat}_H$ and $\text{nat}_{f(H)}$ be the natural homomorphism of $G$ onto $G/H$ and $G_1$ onto $G_1/f(H)$ respectively (Figure 7). Under what condition can we define a homomorphism $h$ of $G/H$ onto $G_1/f(H)$ ?

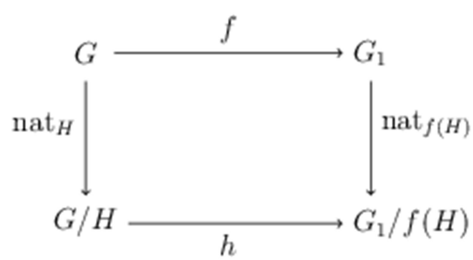


**FIGURE 7**

Here there are four different groups with four binary operations, namely $(G,*)$, $(G_1,\circ)$, $(G/H,\otimes)$ and $(f(G)/f(H),\boxplus)$.

Since $f$ is onto, given any $g_1 \in G$ there is an $a \in G$ such that $g_1 = f(a)$. Then every element in $G_1/f(H)$ is of the form $f(H) \circ f(a)$. We may then define

$$h(H * a) = f(H) \circ f(a).$$

We need to show the mapping is well defined and a homomorphism.

We check it is well defined. Suppose $H * a = H * b$. Then $a * b^{-1} \in H$. This implies $f(a) \circ f(b)^{-1} = f(a * b^{-1}) \in f(H)$ so that $f(H) \circ f(a) = f(H) \circ f(b)$.

For any $H * a$ and $H * b$ in $G/H$,

$$\begin{aligned} h(H * a \otimes H * b) &= h(H * a * b) \\ &= f(H) \circ f(a * b) \\ &= f(H) \circ f(a) \circ f(b) \\ &= f(H) \circ f(a) \boxplus f(H) \circ f(b) \\ &= h(H * a) \boxplus h(H * b) \end{aligned}$$

so it is a homomorphism.

It is also onto since every element of $G_1/f(H)$ is of the form $f(H) \circ g$ and as $f$ is an onto homomorphism, there is an $a \in G$ such that $g = f(a)$ so $f(H) \circ g = f(H) \circ f(a) = h(H * a)$.

Finally if $h$ is one-one then $h$ is an isomorphism. What condition must we impose on $\ker(f)$ and $H$? Suppose $f(H) \circ f(a) = f(H) \circ f(b)$. This implies $f(a * b^{-1}) = f(a) \circ f(b)^{-1} \in f(H)$. Then there is an $h \in H$ such that $f(a * b^{-1}) = f(h)$ which implies $a * b^{-1} * h^{-1} \in \ker(f)$. Therefore if we have $\ker(f) \subseteq H$, then $a * b^{-1} * h^{-1} \in H$, or $a * b^{-1} \in H * h = H$, which then implies $H * a = H * b$.

The diagram in Figure 7 is also commutative because

$$(h \circ \mathrm{nat}_H)(a) = h(Ha) = f(H)f(a) = \mathrm{nat}_{f(H)}f(a) = (\mathrm{nat}_{f(H)} \circ f)(a).$$

We then have

**Theorem 10**. *Let $f$ be a homomorphism of a group $G$ onto a group $G_1$, $H$ be a normal subgroup of $G$ such that $H \supseteq \ker(f)$, and $\mathrm{nat}_H$, $\mathrm{nat}_{f(H)}$ be the natural homomorphisms of $G$ onto $G/H$ and $G_1$ onto $G_1/f(H)$, respectively. Then there exists a unique isomorphism $h$ of $G/H$ onto $G_1/f(H)$ such that $\mathrm{nat}_{f(H)} \circ f = h \circ \mathrm{nat}_H$.*

(Note: The binary operation $\circ$ is function composition between homomorphisms, not the binary operation on $G_1$).

Suppose now instead of $f$, take the identity function $i$ that maps $G$ to $G$. Let $H_1$ and $H_2$ be normal subgroups of $G$. We can define natural homomorphisms $\mathrm{nat}_{H_1}: G \to G_1$ and $\mathrm{nat}_{H_2}: G \to G_2$. We would like to know if there is any homomorphism between $G/H_1$ and $G/H_2$ (Figure 8).

Suppose we define $\phi: G/H_1 \to G/H_2$ by $\phi(H_1 * a) = H_2 * a$. Is $\phi$ well defined? Suppose $H_1 * a = H_1 * b$. Then $a * b^{-1} \in H_1$. If we want $H_2 * a = H_2 * b$, we must impose the condition that $H_1 \subseteq H_2$.

Now $\phi$ is a homomorphism because $\phi(H_1 * a \otimes H_1 * b) = \phi(H_1 * a * b) = H_2 * a * b = H_2 * a \otimes H_2 * b = \phi(H_1 * a) \otimes \phi(H_1 * b)$. It is also onto.

Now by Theorem 8,

$$(G/H_1)/\ker(\phi) \simeq G/H_2.$$

and

$$\begin{aligned}\ker(\phi) &= \{H_1 * a \mid \phi(H_1 * a) = \text{identity of } G/H_2\}\\ &= \{H_1 * a \mid H_2 * a = H_2\}\\ &= \{H_1 * a \mid a \in H_2\}\\ &= H_2/H_1, \text{ the cosets of } H_1 \text{ in } H_2.\end{aligned}$$

With that result, we have the Third Isomorphism Theorem, as a corollary to Theorem 10.

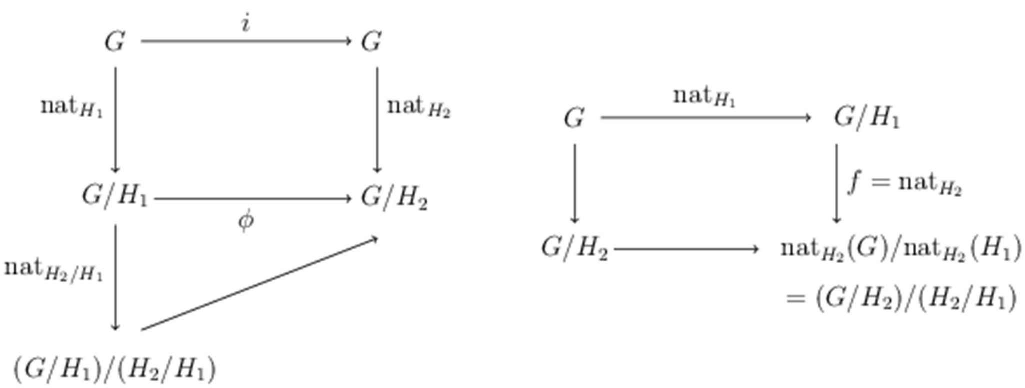


**FIGURE 8**

**Corollary 2** (Third Isomorphism theorem). *Let $H_1, H_2$ be normal subgroups of a group G such that $H_1 \subseteq H_2$. Then*

$$(G/H_1)/(H_2/H_1) \simeq G/H_2.$$

The corollary may also be obtained by taking $f = \text{nat}_{H_1}$ in Theorem 10.

**Example 3**. Consider the group $(\mathbb{Z}, +)$ and the subgroups $\langle 6\rangle$ and $\langle 3\rangle$ of $\mathbb{Z}$ (Figure 9). Then

$$\begin{aligned}Z/\langle 3\rangle &= \{0 + \langle 3\rangle, 1 + \langle 3\rangle, 2 + \langle 3\rangle\}\\ Z/\langle 6\rangle &= \{0 + \langle 6\rangle, 1 + \langle 6\rangle, 2 + \langle 6\rangle, 3 + \langle 6\rangle, 4 + \langle 6\rangle, 5 + \langle 6\rangle\}\\ \langle 3\rangle/\langle 6\rangle &= \{0 + \langle 6\rangle, 3 + \langle 6\rangle\}\end{aligned}$$

and

$$Z/\langle 6\rangle/(\langle 3\rangle/\langle 6\rangle) = \begin{Bmatrix} 0 + \langle 6\rangle + \{0 + \langle 6\rangle, 3 + \langle 6\rangle\}\\ 1 + \langle 6\rangle + \{0 + \langle 6\rangle, 3 + \langle 6\rangle\}\\ 2 + \langle 6\rangle + \{0 + \langle 6\rangle, 3 + \langle 6\rangle\}\end{Bmatrix}.$$

If we identify

$$\begin{aligned}0 + \langle 3\rangle &\leftrightarrow 0 + \langle 6\rangle + \{0 + \langle 6\rangle, 3 + \langle 6\rangle\}\\ 1 + \langle 3\rangle &\leftrightarrow 1 + \langle 6\rangle + \{0 + \langle 6\rangle, 3 + \langle 6\rangle\}\\ 2 + \langle 3\rangle &\leftrightarrow 2 + \langle 6\rangle + \{0 + \langle 6\rangle, 3 + \langle 6\rangle\},\end{aligned}$$

then $Z/\langle 6\rangle/(\langle 3\rangle/\langle 6\rangle)$ is isomorphic to $Z/\langle 3\rangle$.

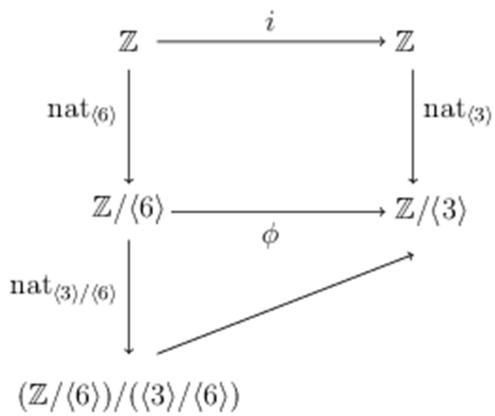


FIGURE 9

## THE CORRESPONDENCE THEOREM

Let $f$ a homomorphism of a group $G$ *onto* a group $G_1$. We notice that every subgroup $H$ of $G$ determines a subgroup $f(H)$ of the group $G_1$. This begs the question, "Is there a one-one correspondence between the subgroups of $G$ and the subgroups of $G_1$?"

Unfortunately, this is not the case. It could happen that distinct subgroups of $G$ may have the same image set in $G_1$. If $H \subseteq M \subseteq H * \ker(f)$, then

$$f(H) \subseteq f(M) \subseteq f\big(H * \ker(f)\big) = f(H) \circ \{e\} = f(H),$$

and this implies f$(H) = f(M)$, that is any subgroup $M$ that lies between $H$ and $H * \ker(f)$, has the same image set as $H$.

We may recover a correspondence if we let $\ker(f) = \{e\}$, or restrict our view to only subgroups $(H,*)$ with $\ker(f) \subseteq H$. In either cases, any subgroup $M$ between $H$ and $H * \ker(f)$, yields the equality $M = H$ because

$$H \subseteq M \subseteq H * \ker(f) \subseteq H.$$

If $\ker(f) = \{e\}$, the homomorphism $f$ is one-to-one, in which case $(G,*)$ and $(G_1,\circ)$ are isomorphic groups. For the second possibility where $\ker(f) \subseteq H$, we have the Correspondence theorem.

We need the following lemmas:

**Lemma 1**. *Suppose $f$ is a homomorphism from $G$ into $G_1$ and $K$ is a subgroup of $G_1$. If $f$ is onto, $f\big(f^{-1}(K)\big) = K$.*

Proof. If $k \in f\big(f^{-1}(K)\big)$ then $k = f(g)$ for some $g \in f^{-1}(K)$, which implies $f(g) \in K$ so that $k = f(g) \in K$. Therefore $f\big(f^{-1}(K)\big) \subseteq K$.

Now suppose $f$ is onto and suppose $k \in K$. Then there is a $g \in G$ such that $f(g) = k \in K$. This implies $g \in f^{-1}(K)$ so that $k = f(g) \in f\big(f^{-1}(K)\big)$. Therefore $K \subseteq f(f^{-1}(K))$ and the lemma is proven.

□

**Lemma 2**. *If $H$ is any subset of $G$ such that* $\ker(f) \subseteq H$, *then* $H = f^{-1}(f(H))$.

Proof. Suppose $a \in H$. Then $f(a) \in f(H)$, and by the definition of inverse image, $a \in f^{-1}(f(H))$. Therefore $H \subseteq f^{-1}(f(H))$. Conversely suppose $a \in f^{-1}(f(H))$. This implies $f(a) \in f(H)$. Then $f(a) = f(h)$ for some $h \in H$. As $f(a) = f(h)$ is equivalent to $f(ah^{-1}) = e_1$ we have $ah^{-1} \in \ker(f) \subseteq H$ or $a \in Hh = H$. This yields the inclusion $f^{-1}(f(H)) \subseteq H$. Therefore $H = f^{-1}(f(H))$.

□

If we consider all the subgroups $H$ of $G$ that contain $\ker(f)$ then we have

**Theorem 11** (Correspondence theorem). *Let $f$ be a homomorphism of a group $G$ onto a group $G_1$. Then $f$ induces a one-one inclusion preserving correspondence between the subgroups $H$ of the group $G$ containing* $\ker(f)$ *and the set of all subgroups $K$ of $G_1$; specifically $K$ is given by $K = f(H)$.*

*In fact, if $H$ and $K$ are corresponding subgroups of $G$ and $G_1$, respectively, then $H$ is a normal subgroup of $G$ if and only if $K$ is a normal subgroup of $G_1$.*

Proof. Let
$$\mathcal{H} = \{H \mid H \text{ is a subgroup of } G \text{ such that } \ker(f) \subseteq H\}$$

and
$$\mathcal{K} = \{K \mid K \text{ is a subgroup of } G_1\}$$

Define $f^*: \mathcal{H} \to \mathcal{K}$ by for all $H \in \mathcal{H}$,
$$f^*(H) = \{ f(h) \mid h \in H \} = f(H).$$

We need to show $f^*$ is a function. Suppose $H_1 = H_2$ be any two subgroups of $G$ containing the kernel of $f$. If $h_1 \in H_1 = H_2$ then $h_1 = h_2$ for some $h_2 \in H_2$ and therefore $f(h_1) = f(h_2)$ because $f$ is well defined. This implies $f(h_1) \in f(H_2)$ and so
$f(H_1) \subseteq f(H_2)$. The reverse inclusion is proven similarly, and we have $f^*(H_1) = f(H_1) = f(H_2) = f^*(H_2)$.

Let $K$ be any subgroup of $G_1$. To show the correspondence is onto we must produce some subgroup $H$ of $G$ with $\ker(f) \subseteq H$ for which $f^*(H) = f(H) = K$. We take $H = f^{-1}(K)$. Then $H$ contains $\ker(f)$ because if $a \in \ker(f)$ then $f(a) = e_1 \in K$ which implies $a \in f^{-1}(K) = H$. Moreover by Lemma 1, $f(H) = f(f^{-1}(K)) = K$.

Next we show the correspondence $f^*$ is one-to-one. Let $K_1$ and $K_2$ be in $\mathcal{K}$ such that $K_1 = K_2$. Since we have shown the correspondence to be onto, there exist $H_1$ and $H_2$ in $\mathcal{H}$ such that $K_1 = f(H_1)$ and $K_2 = f(H_2)$. Now
$$H_1 = f^{-1}(f(H_1)) = f^{-1}(f(H_2)) = H_2,$$

by Lemma 2, and this proves $f^*$ is one-to-one.

Suppose $H$ is a normal subgroup of $G$ such that $\ker(f) \subseteq H$. Let $f^*(H) = K = f(H)$. We show that $K$ is a normal subgroup of $G_1$. Let $f(a) = g \in G_1$ and $f(h) = k \in K$. Then $g * k * g^{-1} = f(a) \circ f(h) \circ f(a)^{-1} = f(a * h * a^{-1}) \in f(H) = K$ because $a * h * a^{-1} \in H$ since $H$ is normal.

Conversely suppose $N$ is a normal subgroup of $G_1$. Since the correspondence is one to one, there is a subgroup $M$ of $G$ with $\ker(f) \subseteq M$ such that $f^*(M) = f(M) = N$. Now for any $g \in G$ and $m \in M$, $f(g * m * g^{-1}) = f(g) \circ f(m) \circ f(g)^{-1} \in f(M)$ since $f(M) = N$ is normal. This implies $f(g * m * g^{-1}) = f(m_1)$ for some $m_1 \in M$ or
$g * m * g^{-1} * m_1^{-1} \in \ker(f) \subseteq M$. Then $g * m * g^{-1} = Mm_1 = M$ which proves that $M$ is a normal subgroup of $G$.

□

Now let $N$ be a normal subgroup of $G$ and $\text{nat}_N: G \to G/N$ be the canonical mapping. By Theorem 5, the kernel of $\text{nat}_N$ is $N$. By Theorem 11, there is a one-one correspondence between the subgroups of $G$ containing $N$ and the subgroups of $G/N$.

Given any subgroup $H$ of $G/N$,

$$\begin{aligned} H &= \text{nat}_N^*(K), \text{ for some subgroup } K \text{ of } G \\ &= \{\, \text{nat}_N(a) \mid a \in K \,\} \\ &= \{\, N * a \mid a \in K \,\} \\ &= K/N. \end{aligned}$$

Furthermore if $H = G/K$ is a normal subgroup of $G/N$ then $K$ is a normal subgroup of $G$. Let $k \in K$ and $g \in G$. Then $N * (g * k * g^{-1}) = (N * g) \otimes (N * k) \otimes (N * g)^{-1}$
$= N * k_1$ for some $k_1 \in K$. This implies $g * k * g^{-1} * k_1^{-1} \in N \subseteq K$ and therefore $g * k * g^{-1} \in K * k_1 = K$.

Conversely if $K$ is a normal subgroup of $G$, for any $g \in G$ and $k \in K$, $(N * g) \otimes (N * k) \otimes (N * g)^{-1} = N * g * k * g^{-1} = N * k_1 \in K/N$ for some $k_1 \in K$. The proves $K/N$ is a normal subgroup of $G/N$.

We obtain as a corollary (Figure 10):

**Corollary 3**. *Let $N$ be a normal subgroup of a group $G$. Then every subgroup of $G/N$ is of the form $K/N$, where $K$ is a subgroup of $G$ that contains $N$. Also, $K/N$ is a normal subgroup of $G/N$ if and only if $K$ is a normal subgroup of $G$.*

**Example 4**. Let $f$ be a homomorphism of $(\mathbb{Z}, +)$ onto $(\mathbb{Z}_{12}, +_{12})$ defined by $f(n) = [n]$, for all $n \in \mathbb{Z}$ (Figure 11). The kernel of $f$ is $\langle 12 \rangle = \{12n \mid n \in \mathbb{Z}\}$. The subgroups of $(\mathbb{Z}, +)$ containing $\langle 12 \rangle$ are

$$\mathcal{H} = \{\langle 12 \rangle, \langle 6 \rangle, \langle 4 \rangle, \langle 3 \rangle, \langle 2 \rangle, \mathbb{Z}\}$$

The subgroups of $(\mathbb{Z}_{12}, +_{12})$ are

$$\mathcal{K} = \{[0], [6], [4], [3], [2], \mathbb{Z}_{12}\}.$$

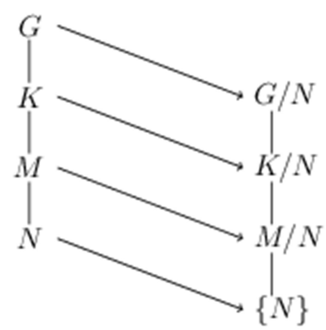


**FIGURE 10**

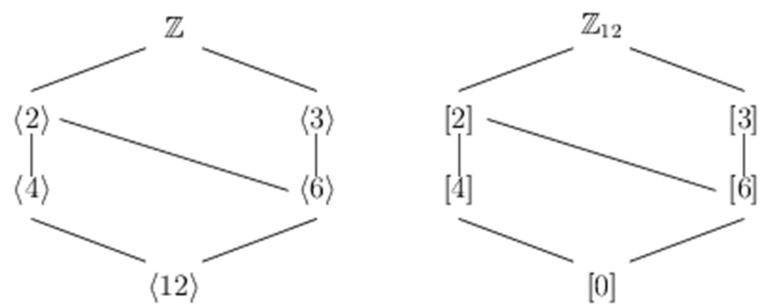


**FIGURE 11**

The one-to-one correspondence are

$$f*: \langle 12\rangle \to [0], \quad f*: \langle 3\rangle \to [3],$$
$$f*: \langle 2\rangle \to [2], \quad f*: \langle 6\rangle \to [6],$$
$$f*: \langle 4\rangle \to [4], \quad f*: \mathbb{Z} \to \mathbb{Z}_{12}.$$

## CONCLUSION

The isomorphism theorems presented here share a common characteristic, which is given by the Fundamental Theorem. We start with a homomorphism $f$, determine the kernel of $f$ and obtain an isomorphism $h$ from the group $G/\ker(f)$ to the image of $f$. The Fundamental Theorem is also known as the Factor Theorem since the homomorphism $f$ can be "factored" as $h \circ \text{nat}_H$.

It is hoped that the discussion on isomorphisms may inspire the reader to explore a more challenging topics in Group Theory such as the Jordan-Holder theorem and the Sylow theorems.

## BIBLIOGRAPHY

Burton, D. M. (1967), *Introduction to modern abstract algebra*. Addison-Wesley Pub. Co.

Malik, D. S,, Mordeson, J. M., Sen, M. K. (1997), *Fundamentals of abstract algebra*. McGraw-Hill.