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New Third Order Convergence Iterative Method for Finding Multiple Roots of Nonlinear Equations

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ABSTRACT

In this paper, we derive a new modified third order convergence iterative methods for computing multiple roots of non-linear equations. Our proposed scheme requires one evaluation of function and two evaluations of first derivative. Error term is proved to possess a third order. Numerical performance shows that the proposed method provides a highest accuracy results as compared to other existing third order iterative methods.

Keywords: Multi-point iterative methods, Multiple roots, Nonlinear equations, Order of convergence.

INTRODUCTION

There are vast literatures on the solution of nonlinear equations. The well known modified Newton's method for multiple roots is given by Schroeder and Stewart (1998)

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}, \tag{1}$$

which is quadratically converges.

The example of one-point third order convergence iterative method for multiple zeros is developed by Hansen and Patrick (1976) given by

$$\hat{x}_{k+1} = x - \frac{m(\nu+1)f(x)}{\nu f'(x) \pm \sqrt{m(\nu-\nu)f'(x)^2 - m(\nu+1)f(x)f''(x)}}. \tag{2}$$

Osada (1994) obtained the new third order iterative method for computing multiple zeros as

$$x_{k+1} = x_k - \frac{1}{2}m(m+1)\frac{f(x_k)}{f'(x_k)} + \frac{1}{2}(m-1)^2\frac{f'(x_k)}{f''(x_k)}, \tag{3}$$

which requires one evaluation of function, f, one evaluation of first derivative function, f' and one evaluation of second derivative function, f''. By combining Osada's method (3) and Euler-

Chebyshev's method (Traub, 1977), Chun et al. (2009) produced the new iterative method for multiple zeros given by

$$x_{k+1} = x_k - \frac{m\big((2\gamma-1)m+3-2\gamma\big)}{2}\frac{f(x_k)}{f'(x_k)} + \frac{\gamma(m-1)^2}{2}\frac{f'(x_k)}{f''(x_k)} - \frac{m^2(1-\gamma)}{2}\frac{f(x_k)^2 f''(x_k)}{(f'(x_k))^3} \tag{4}$$

where,

$$A = \mu^{2m} - \mu^{m+1}, B = -\frac{\mu^m(m-2)(m-1)+1}{(m-1)^2}, \mu = \frac{m}{m-1}, \gamma = -1 \text{ or } \frac{1}{2}.$$

Victory and Neta (1983) proposed the third order convergence method which requires two evaluations of function and one evaluation of first derivative of function written as

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_k - \dfrac{f(y_k)}{f'(x_k)} \cdot \dfrac{f(x_k)+Af(y_k)}{f(x_k)+Bf(y_k)}, \end{cases} \tag{5}$$

where $A = \mu^{2m} - \mu^{m+1}, B = -\dfrac{\mu^m(m-2)(m-1)+1}{(m-1)^2}$ and $\mu = \dfrac{m}{m-1}$. Equation (5) is an example of two-point method. Neta (2008) developed the new iterative method for multiple zeros with the same number of function evaluations as in (5), written as

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_k - \dfrac{f(x_k)}{f'(x_k)}\left(\beta + \gamma\dfrac{f(y_k)}{f'(x_k)}\right), \end{cases} \tag{6}$$

where,

$$\beta = m - \frac{m(m-\theta)}{\theta^2}, \gamma = \frac{m(m-\theta)}{\rho\theta^2}, \rho = \left(\frac{m-\theta}{m}\right)^m \text{ and } \theta \in \mathbb{R}.$$

In this paper, we develop a new third order iterative method for finding multiple roots of nonlinear equation with known multiplicity, m and free from second derivative functions.

CONSTRUCTION OF METHODS

The well-known third order Halley's method (Petkovic et al., 2012) for simple zeros is given by

$$x_{k+1} = x_k - \frac{2f(x_k)f'(x_k)}{2(f'(x_k))^2 - f(x_k)f''(x_k)}. \tag{7}$$

Let the Newton-type iterative method for multiple zeros be

$$y_k = x_k - \frac{2m}{m+2}\frac{f(x_k)}{f'(x_k)}. \tag{8}$$

Expand the function $f'(y_k)$ in (8) about $x = x_k$, we obtain

$$f''(x_k) \cong \frac{(m+2)f'(x_k)\left[f'(x_k) - f'(y_k)\right]}{2mf(x_k)}. \tag{9}$$

Substitute (9) into (7) yield

$$\begin{cases} y_k = x_k - \dfrac{2m}{m+2}\dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{4mf(x_k)}{(3m-2)f'(x_k) + (m+2)f'(y_k)}. \end{cases} \tag{10}$$

In order to archive the third order convergence, we assign the free disposable parameters α and β. Then (10) reads

$$\begin{cases} y_k = x_k - \dfrac{2m}{m+2}\dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{4\alpha mf(x_k)}{\beta(3m-2)f'(x_k) + (m+2)f'(y_k)}. \end{cases} \tag{11}$$

CONVERGENCE ANALYSIS

Theorem 1. *Let D be an open interval and $x^* \in D$ be a multiple zeros of a sufficiently smooth function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ with the multiplicity $m > 1$, which includes x_0 as an initial approximation of x^*. Then, the iterative method defined by (11) has third order convergence when*

$$\begin{cases} \alpha = m^{-2+m}(2+m)^{1-m}, \\ \beta = m^{-3+m}(2+m)^{-m}(8 - m^2(4+m) \end{cases}$$

and the error term is

$$e_{k+1} = -\frac{(-2+m)c_1^2 e_n^3}{m^3} + O(e_k)^4.$$

Proof. Let $e_k := x_k - x^*, e_k := y_k - x^*, c_i := \dfrac{m!}{(m+i)!}\dfrac{f^{m+i}(x^*)}{f^m(x^*)}, c_0 = 1, p = m+1, q = m+2, r = m-1$ and $g = \dfrac{-1+rm}{m^2}$. Since $f(x^*) = 0$, Taylor expansion of f at x^* yields

$$f(x_k) = e_k^m\left(1 + c_1 e_k + c_2 e_k^2 + c_3 e_k^3\right) + O(e_k^4), \tag{12}$$

and

$$f'(x_k) = e^r\left(m + e_n pc_1 + e_n^2 pc_2 + e_n^3\left((m+3)c_3 + O(e_n^4)\right)\right), \tag{13}$$

hence

$$e_{k,y} = y_k - x^* = \frac{me_k}{q} + \frac{2c_1e_k^2}{mq} + \frac{\left(-2pc_1^2 + 4mc_2\right)e_k^3}{m^2q} + O(e_k^4). \tag{14}$$

For $f(y_k)$ we have

$$f(y_k) = e_{n,y}^m\left(1 + c_1e_{n,y} + c_2e_{n,y}^2 + c_3e_{n,y}^3\right) + O(e_{n,y}^4). \tag{15}$$

Substituting (12)-(15) in (11), we obtain

$$e_{n+1} = D_1e_n + D_2e_n^2 + D_3e_n^3 + O(e_n^4),$$

where

$$D_1 = 1 - \frac{4\alpha m}{m^m q^{2-m} + \beta m^2}, \tag{16}$$

$$D_2 = \frac{4q^m\alpha\left(m^m\left(-8 + m^2(m+4)\right) + m^3q^m\beta\right)c_1}{m\left(m^mq^2 + m^2q^m\beta\right)^2}, \tag{17}$$

$$D_3 = -\frac{\left(4q^m\alpha Tc_1^2 - 2m\left(m^mq^2 + m^2q^m\beta\right)\left(Um^mc_2\right)\right)}{\left(m^2\left(m^mq^2 + m^2q^m\beta\right)^3\right)}, \tag{18}$$

$$T = m^{2m}q^2(-16 + m(-8 + mp(4+m))) + 2m^pq^p\left(-4 - 2m + 3m^3 + m^4\right)\beta + m^5pq^{2m}\beta^2$$

and

$$U = \left(-8 + m^2(4+m)\right) + m^3q^m\beta.$$

In order to obtain the third order convergence, it is necessary to choose

$D_i = 0 \ \ (i = 1,2)$, which give

$$\alpha = m^{m-2}(m+2)^{1-m}$$

and

$$\beta = m^{m-3}(m+2)^{-m}\left(8 - m^2(m+4)\right).$$

The error term becomes

$$e_{n+1} = -\frac{(-2+m)c_1^2e_n^3}{m^3} + O(e_n)^4,$$

which completes the proof.

NUMERICAL ANALYSIS

Numerical tests are carried out to see the performance of the newly modified methods as compared to other third order convergence methods. The test functions used in our computational works are listed in Table 1. Software package Mathematica 11 with 200 significant digit multi-precision is used. We compute the error bound (EB), the computational order of convergence (COC) (Weerakoon and Fernando, 2000) and the approximated computational order of convergence (ACOC) (Grau-Sanchez et al, 2010), which are defined respectively as

$$EB = |x_k - x^*|, \tag{19}$$

$$COC \approx \frac{\ln|(x_{n+1}-x^*)/(x_n-x^*)|}{\ln|(x_n-x^*)/(x_{n-1}-x^*)|} \tag{20}$$

and

$$ACOC \approx \frac{\ln\dfrac{|(x_{n+1}-x_n)|}{|(x_n-x_{n-1})|}}{\ln\dfrac{|(x_n-x_{n-1})|}{|(x_{n-1}-x_{n-2})|}}. \tag{21}$$

We compare our methods with the following existing methods :

1. *Bodewig's method or Halley-like method* (BM) (Petkovic et al., 2012), given by

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{m+1}{2m}f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}}. \tag{22}$$

2. *Dong's method* (DM) (Dong, 1987) given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)}{\left(\dfrac{m}{m-1}\right)^{m+1} f'(y_n) + \dfrac{m-m^2-1}{(m-1)^2} f'(x_n)}. \end{cases} \tag{23}$$

3. *Ferrara et al.'s method* (FSS) (Ferrara, 2015), given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{\theta f(x_n)}{\theta f(x_n) - f(y_n)} \dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{24}$$

where $\theta = \left(\dfrac{-1+m}{m}\right)^{-1+m}$.

4. *Hommier's method* (HM) (Hommier, 2009), given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - m^2\left(\dfrac{m}{m+1}\right)^{m-1}\dfrac{f(x_n)}{f'(x_n)} + m(m-1)\dfrac{f(x_n)}{f'(x_n)}. \end{cases} \quad (25)$$

Table 1 : List of test Functions

Test function f_n	**Root** x^*	**Multiplicity** m
$f_1(x) = (\sin^2 x + x)^5$	0	5
$f_2(x) = (\ln(1+x^2) + e^{x^2-3x}\sin x)^6$	0	6
$f_3(x) = (x^3 + \ln(1+x))^7$	0	7
$f_4(x) = (x^6-8)^2\ln(x^6-7)$	$\sqrt{2}$	3
$f_5(x) = (\ln(x^3-x+1) + 4\sin x - 1)^{10}$	1	10

Table 2 : Error, COC and ACOC of Method, DM, BM, HM and FSS

Methods	**Method (11)**	**BM (22)**	**DM (23)**	**FSS (24)**	**HM (25)**
$f_1, x_0 = 0.1$					
$\lvert x_1 - x^*\rvert$	$0.270e^{-3}$	$0.820e^{-3}$	$0.420e^{-3}$	$0.740e^{-3}$	$0.115e^{-2}$
$\lvert x_2 - x^*\rvert$	$0.118e^{-10}$	$0.550e^{-9}$	$0.314e^{-10}$	$0.364e^{-9}$	$0.242e^{-8}$
$\lvert x_3 - x^*\rvert$	$0.996e^{-33}$	$0.167e^{-27}$	$0.132e^{-31}$	$0.434e^{-28}$	$0.227e^{-25}$
$\lvert x_4 - x^*\rvert$	$0.593e^{-99}$	$0.462e^{-83}$	$0.969e^{-96}$	$0.787e^{-85}$	$0.189e^{-76}$
COC	3.0000	3.0000	3.0000	3.0000	3.0000
ACOC	3.0001	2.9999	3.0000	2.9999	2.9997
$f_2, x_0 = 0.3$					
$\lvert x_1 - x^*\rvert$	$0.124e^{-1}$	$0.645e^{-1}$	$0.479e^{-1}$	$0.564e^{-1}$	$0.503e^{-1}$
$\lvert x_2 - x^*\rvert$	$0.546e^{-5}$	$0.484e^{-4}$	$0.116e^{-3}$	$0.178e^{-4}$	$0.478e^{-3}$
$\lvert x_3 - x^*\rvert$	$0.435e^{-15}$	$0.151e^{-12}$	$0.223e^{-11}$	$0.437e^{-14}$	$0.300e^{-9}$
$\lvert x_4 - x^*\rvert$	$0.219e^{-45}$	$0.458e^{-38}$	$0.158e^{-34}$	$0.651e^{-43}$	$0.738e^{-28}$
COC	3.0000	3.0000	3.0001	3.0000	3.0001
ACOC	3.0098	2.7218	2.9490	2.7447	3.0735
$f_3, x_0 = 0.2$					
$\lvert x_1 - x^*\rvert$	$0.650e^{-3}$	$0.108e^{-1}$	$0.781e^{-2}$	$0.925e^{-2}$	$0.797e^{-2}$
$\lvert x_2 - x^*\rvert$	$0.495e^{-10}$	$0.132e^{-5}$	$0.376e^{-6}$	$0.702e^{-6}$	$0.283e^{-6}$
$\lvert x_3 - x^*\rvert$	$0.217e^{-31}$	$0.251e^{-17}$	$0.425e^{-19}$	$0.316e^{-18}$	0.134^{-19}
$\lvert x_4 - x^*\rvert$	$0.182e^{-95}$	$0.172e^{-52}$	$0.248e^{-52}$	$0.286e^{-55}$	$0.144e^{-59}$

COC	3.0000	3.0000	3.0000	3.0000	3.0000
ACOC	3.0007	2.9967	2.9986	2.9972	2.9936
$f_4, x_0 = 1.5$					
$\lvert x_1 - x^* \rvert$	$0.991e^{-3}$	$0.423e^{-2}$	$0.221e^{-2}$	$0.329e^{-2}$	$0.299e^{-2}$
$\lvert x_2 - x^* \rvert$	$0.522e^{-8}$	$0.599e^{-5}$	$0.329e^{-6}$	$0.163e^{-5}$	$0.631e^{-6}$
$\lvert x_3 - x^* \rvert$	0.716^{-24}	$0.134e^{-13}$	$0.970e^{-18}$	$0.171e^{-15}$	$0.551e^{-17}$
$\lvert x_4 - x^* \rvert$	0.185^{-71}	0.151^{-39}	0.613^{-58}	0.195^{-45}	0.368^{-50}
COC	3.0000	3.0000	3.0000	3.0000	3.0000
ACOC	3.0050	3.0357	3.0134	3.0207	3.0088
$f_5, x_0 = 1.2$					
$\lvert x_1 - x^* \rvert$	$0.551e^{-4}$	$0.181e^{-2}$	$0.146e^{-2}$	$0.164e^{-2}$	$0.152e^{-2}$
$\lvert x_2 - x^* \rvert$	$0.134e^{-14}$	$0.163e^{-8}$	$0.688e^{-9}$	$0.109e^{-8}$	$0.832e^{-9}$
$\lvert x_3 - x^* \rvert$	$0.192e^{-46}$	$0119e^{-26}$	$0.719e^{-28}$	$0.322e^{-27}$	$0.137e^{-27}$
$\lvert x_4 - x^* \rvert$	$0.569e^{-142}$	$0.465e^{-81}$	$0.820e^{-85}$	$0.853e^{-83}$	$0.618e^{-84}$
COC	3.0000	3.0000	3.0000	3.0000	3.0000
ACOC.	2.9999	2.9999	2.9999	2.9999	2.9998

Table 2 shows that our proposed method (11) provides the smaller error per iteration as compared to others, which mean our method gives highest accuracy and converge faster. The value of COC and ACOC of our proposed method proves that it possess as third order convergence.

CONCLUSION

In this work, we have derived a new modified third order convergence iterative method for solving multiple roots of nonlinear equation. The proposed methods is free from second derivative of function. Numerical results show that our method gives faster convergence and smaller error per iteration as compared to others with the same order of convergence.

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