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The Conjugacy Classes of Some Three-Generator Groups

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ABSTRACT

The conjugacy class of an element in a group is the set of all conjugates of that element in the group. An element x is conjugate to y in a group G if there exists an element g in G such that $g^{-1}xg = y$. Many of previous researchers only found the lower bound of the conjugacy classes without giving the exact number of conjugacy classes. In this paper, the number of the conjugacy classes of 3-generator groups of order 16 is computed.

Keywords: Conjugacy class; Groups of order 16.

INTRODUCTION

The conjugacy class of a group is sometimes viewed as a partition of a group. Let a be an element of a group G. The conjugacy class of the element a is defined by the following; Let b be an element in G. Then a and b are conjugate if $x^{-1}ax = b$ for some x in G. The conjugacy class of a is written as cl(a) = $\{x^{-1}ax | x\epsilon\ G\}$. In recent years, a number of researchers have studied on the bounds on the number of conjugacy classes. For any non-trivial group G, the number of conjugacy classes, $\text{cl}_G > 1$. In 1968, Erdos and Turan stated that the number of conjugacy classes can be calculated using the formula $\text{cl}_G > log_2 log_2 |G|$. In the same year, Polan (1968) has made an improvement on the bound of cl_G for nilpotent group which is $\text{cl}_G > log_2$. Besides, Sherman (1979) also investigated the best lower bound for the conjugacy classes. As for the case that if G is a nilpotent group, Sherman revealed that $\text{cl}_G > c(|G|^{1/c} - 1) + 1$ where c is the class of nilpotent group.

However, the improvement made by Sherman (1979) was still weak. He gave a better estimation on the bounds of the conjugacy classes; however it is still far away from the exact value. In 2008, Ahmad found the exact number of conjugacy classes for 2-generator p-groups of class 2. The scope of this paper is on the 3-generator groups of order 16. There are three groups under this category, H_1, H_2, and H_3 given by the following presentations (S.Ok, 2001):

$H_1 = \langle x, y, z \mid x^2 = y^2 = z^4 = 1, [x,z] = [y,z] = 1, [x,y] = z^2 \rangle$,

$H_2 = \langle x, y, z \mid x^4 = y^2 = z^2 = 1, [x,z] = [y,z] = 1, x^y = x^3 \rangle$,

$H_3 = \langle x, y, z \mid x^4 = y^2 = z^2 = 1, [x, y] = z, [x,z] = [y, z] = 1 \rangle$.

This paper is structured as follows: The first section is the introduction section. It includes basic definitions and concepts in group theory that are used in this research.

PRELIMINARIES

In this section, some basic properties which are needed later are presented.

Proposition 1.1 (Fraleigh, 2000) The conjugacy class of the identity element is its own class, namely cl(1) = {1}.

Proposition 1.2 (Fraleigh, 2000) Let a and b be two elements in a finite group G. The elements a and b are conjugate if they belong in one conjugacy class, that is cl(a) = cl(b).

Proposition 1.3 (Fraleigh, 2000) Suppose a is an element of a group G, then a lies in the center $Z(G)$ if and only if its conjugacy class has only one element.

MAIN RESULTS

In this section, the number of conjugacy classes of all three generator groups of order 16 are computed and determined. The number of conjugacy classes of a group G is denoted by K(G).

Theorem 2.1 Let $H_1 = \langle x, y, z \mid x^2 = y^2 = z^4 = 1, [x, z] = [y, z] = 1, [x, y] = z^2 \rangle$. Then the number of conjugacy classes of H_1, $K(H_1) = 10$.

Proof: Using definition of conjugacy class, the conjugacy classes of H_1 can be found. Based on the definition, cl(a) = $\{g \in G$: there exists $x \in G$ with $g = x^{-1}ax\}$. Thus, the conjugacy classes of H_1 are determined as follows: First, let a = 1. Since the conjugacy class of the identity element is the identity itself, then, cl(1) = {1}.

Next, we find the conjugacy class for a, cl(a) = $\{ga\,g^{-1}\}$ where $g \in \{1, x, y, z, z^2, z^3, xy, xz, xz^2, xz^3, yz, yz^2, yz^3, xyz, xyz^2, xyz^3\}$ for all a in H_1. The conjugate elements of a are determined as follows:

When $g = x$, $gag^{-1} = (x)x(x)^{-1} = xxx = x$,
when $g = y$, $gag^{-1} = (y)x(y)^{-1} = yxy = yyxz^2 = xz^2$,
when $g = z$, $gag^{-1} = (z)x(z)^{-1} = zxz = zxz^3 = xzz^3 = x$,
when $g = z^2$, $gag^{-1} = (z^2)x(z^2)^{-1} = z^2xz^2 = zzxzz = zxzzz = xzzzz = x$,
when $g = z^3$, $gag^{-1} = (z^3)x(z^3)^{-1} = z^3xz = z^3zx = z^4x = x$,
when $g = xy$, $gag^{-1} = (xy)x(xy)^{-1} = (xy)x(xyz^2) = xyxxyz^2 = xyyz^2 = xz^2$,
when $g = xz$, $gag^{-1} = (xz)x(xz)^{-1} = (xz)x(xz^3) = xzxxz^3 = x$,
when $g = xz^2$, $gag^{-1} = (xz^2)x(xz^2)^{-1} = (xz^2)x(xz^2) = xz^2xxz^2 = x$,
when $g = xz^3$, $gag^{-1} = (xz^3)x(xz^3)^{-1} = (xz^3)x(xz) = xz^3xxz = x$,
when $g = yz$, $gag^{-1} = (yz)x(yz)^{-1} = (yz)x(yz^3) = yzxyz^3 = yzyxz^2z^3 = yyzxz = xzz = xz^2$,
when $g = yz^2$, $gag^{-1} = (yz^2)x(yz^2)^{-1} = (yz^2)x(yz^2) = yz^2xyz^2 = yz^2yxz^3z^2 = yz^2yx = yzzyx = zzx = z^2x = zzx = zxz = xzz = xz^2$,
when $g = yz^3$, $gag^{-1} = (yz^3)x(yz^3)^{-1} = (yz^3)x(yz) = yz^3xyz = yz^3yxz^2z = yzzzyxz^2z = yzzyzxz^3 = yzzyzxz^3 = yzyzzxz^3 = zzzxz^3 = z^3xxz^2 = xz^2$,
when $g = xyz$, $gag^{-1} = (xyz)x(xyz)^{-1} = (xyz)x(xyz) = xyzyz = xzz = xz^2$,
when $g = xyz^2$, $gag^{-1} = (xyz^2)x(xyz^2)^{-1} = (xyz^2)x(xy) = xyz^2y = xyzzy = xzyzy = xzz = xz^2$,
when $g = xyz^3$, $gag^{-1} = (xyz^3)x(xyz^3)^{-1} = (xyz^3)x(xy\ ^3) = xyz^3yz^3 = xyz^2zyz^3 = xyz^2y = xyzzy = xzyzy = xzz = xz^2$.

Thus, cl(x) = $\{x, xz^2\}$. Since x and xz^2 belong in one conjugacy class, then, cl(x) = cl(xz^2).

By replacing g with the rest of the elements in the set $\{y, z, z^2, z^3, xy, xz, xz^2, xz^3, yz, yz^2, yz^3, xyz, xyz^2, xyz^3\}$, the conjugacy classes of H_1 are found as follows:

(i) cl(1) = {1},
(ii) cl(x) = $\{x, xz^2\}$ = cl(xz^2),
(iii) cl(y) = $\{y, yz^2\}$ = cl(yz^2),
(iv) cl(z) = $\{z\}$,
(v) cl(z^2) = $\{z^2\}$,
(vi) cl(z^3) = $\{z^3\}$,
(vii) cl(xy) = $\{xy, xyz^2\}$ = cl(xyz^2),
(viii) cl(xz) = $\{xz, xz^3\}$ = cl(xz^3),
(ix) cl(yz) = $\{yz, yz^3\}$ = cl(yz^3),
(x) cl(xyz) = $\{xyz, xyz^3\}$ = cl(xyz^3).

Therefore, $K(H_1) = 10$. ■

Theorem 2.2 Let $H_2 = \langle x, y, z \mid x^4 = y^2 = z^2 = 1, [x, z] = [y, z] = 1, x^y = x^3 \rangle$. Then the number of conjugacy class of H_2, $K(H_2) = 10$.

Proof: The conjugacy classes of H_2 are computed in a similar way for H_1. The conjugacy classes of H_2 are calculated as follows:

For all a in H_2, cl(a) = $\{gag^{-1}\}$ where $g \in \{1, x, y, z, z^2, z^3, xy, xz, xz^2, xz^3, yz, yz^2, yz^3, xyz, xyz^2, xyz^3\}$. The conjugate elements of a are determined as follows:

When $g = x$, $gag^{-1} = (x)x(x)^{-1} = xxx^3 = x$,
when $g = x^2$, $gag^{-1} = (x^2)x(x^2)^{-1} = x^2xx^2 = x$,
when $g = x^3$, $gag^{-1} = (x^3)x(x^3)^{-1} = x^3xx = x$,
when $g = y$, $gag^{-1} = (y)x(y)^{-1} = yxy = yyx^3 = x^3$,
when $g = z$, $gag^{-1} = (z)x(z)^{-1} = zxz = zzx = x$,
when $g = xy$, $gag^{-1} = (xy)x(xy)^{-1} = xyxxy = yx^3xxy = yxy = yyx^3 = x^3$,
when $g = x^2y$, $gag^{-1} = (x^2y)x(x^2y)^{-1} = xxyxxxy = xyx^3xxxy = xyxxy = yx^3xxy = yxy = yyx^3 = x^3$,
when $g = x^3y$, $gag^{-1} = (x^3y)x(x^3y)^{-1} = (xy)wxy(xy) = x^3yxx^3y = x^3$,
when $g = xz$, $gag^{-1} = (xz)x(xz)^{-1} = xzxx^3z = x$,
when $g = x^2z$, $gag^{-1} = (x^2z)x(x^2z)^{-1} = xxzxxxz = xzxxxxz = x$,
when $g = x^3z$, $gag^{-1} = (x^3z)x(x^3z)^{-1} = (x^3z)x(xz) = x^3xzzx = x$,
when $g = yz$, $gag^{-1} = (yz)x(yz)^{-1} = (yz)x(yz) = zyxyz = zyyx^3z = zxxxz = xzxzx = xzzxx = x^3$,
when $g = xyz$, $gag^{-1} = (xyz)x(xyz)^{-1} = xyzxxyz = xyxzxyz = yx^3xzxyz = yzxyz = zyyx^3z = zxxxz = xzxzx = xzzxx = x^3$,
when $g = x^2yz$, $gag^{-1} = (x^2yz)x(x^2yz)^{-1} = x^2yzxx^2yz = xxyxzxxyz = xyx^3xzxxyz = xyxzxyz = yx^3xzxyz = yzxyz = zyyx^3z = zxxxz = xzxzx = xxzzx = x^3$,
when $g = x^3yz$, $gag^{-1} = (x^3yz)x(x^3yz)^{-1} = x^2yzxx^3yz = x^3$.

Since cl(x) = $\{x, x^3\}$, then cl(x) = cl(x^3).

The same steps are used for $g = \{y, z, z^2, z^3, xy, xz, xz^2, xz^3, yz, yz^2, yz^3, xyz, xyz^2, xyz^3\}$. Therefore, the conjugacy classes of H_2 are listed as follows:

(i) cl(1) = {1},

(ii) cl(x) = {x, x^3} = cl(x^3),
(iii) cl(x^2) = {x^2},
(iv) cl(xy) = {xy, x^3y} = cl(x^3y),
(v) cl(xz) = {xz, x^3z} = cl(x^3z),
(vi) cl(y) = {y, x^2y} = cl(x^2y),
(vii) cl(z) ={z},
(viii) cl(x^2z) = {x^2z},
(ix) cl(yz) = {yz, x^2yz} = cl(x^2yz),
(x) cl(xyz) = {xyz, x^3yz} = cl(x^3yz).

Thus, $K(H_2)$ = 10. ■

Theorem 2.3 Let H_3= <x, y, z | $x^4= y^2= z^2$= 1, [x, y] = z, [x, z] = [y, z] = 1 >. Then the number of conjugacy class of H_3, $K(H_3)$ = 10.

Proof: By using the same steps as in the Theorem 2.1 and Theorem 2.2, the conjugacy classes of H_3 are obtained. Therefore, the conjugacy classes of H_3 are computed as follows:

Using the definition, cl(a) = { $ga\, g^{-1}$} where $g \in$ {1, x, y, z, z^2, z^3, xy, xz, xz^2, xz^3, yz, yz^2, yz^3, xyz, xyz^2, xyz^3}. The conjugate elements of a are determined as follows:

When $g = x$, $gag^{-1} = (x)x(x)^{-1} = xxx^3 = x$,
when $g = x^2$, $gag^{-1} = (x^2)x(x^2)^{-1} = x^2xx^2 = x$,
when $g = x^3$, $gag^{-1} = (x^3)x(x^3)^{-1} = x^3xx = x$,
when $g = y$, $gag^{-1} = (y)x(y)^{-1} = yxy = yyxz = xz$,
when $g = z$, $gag^{-1} = (z)x(z)^{-1} = zxz = xzz = x$,
when $g = xy$, $gag^{-1} = (xy)x(xy)^{-1} = xyx(x^3yz) = xyxx^3yz = xyyz = xz$,
when $g = x^2y$, $gag^{-1} = (x^2y)\, x(x^2y)^{-1} = x^2yxx^2y = xxyxxxy = xyxzxxxy = xyzxxxxy = xyzy = xzyy = xz$,
when $g = x^3y$, $gag^{-1} = (x^3y)x(x^3y)^{-1} = (x^3y)x(xyz) = x^3yxxyz = x^2xyxxyz = x^2yxzxxyz = xxyzxxxyz = xyxzzxxxyz = xz$,
when $g = xz$, $gag^{-1} = (xz)x(xz)^{-1} = xzxx^3z = x$,
when $g = x^2z$, $gag^{-1} = (x^2z)x(x^2z)^{-1} = x^2zxx^2z = xxzxx^2z = xzxxx^2z = x$,
when $g = x^3z$, $gag^{-1} = (x^3z)x(x^3z)^{-1} = (x^3z)x(xz) = x^3xzxz = zxz = xzz = x$,
when $g = yz$, $gag^{-1} = (yz)x(yz)^{-1} = (yz)x(yz) = yzyxzz = yzyx = zyyx = xz$,
when $g = xyz$, $gag^{-1} = (xyz)x(xyz)^{-1} = xyzxx^3y = xyzy = xyyz = xz$,
when $g = x^2yz$, $gag^{-1} = (x^2yz)x(x^2yz)^{-1} = x^2yzxx^2yz = xxyzx\ {}^2yz = xyxzzxx^2yz = xz$,
when $g = x^3yz$, $gag^{-1} = (x^3yz)x(x^3yz)^{-1} = x^3yzxxy = x^3yzxyxz = x^3yzyxzxz = x^3yzyxzzx = x^3yzyxx = x^3zyyxx = x^3zxx = x^3xzx = xz$.

Since cl(x) = {x, xz}, then cl(x) = cl(xz).

The calculations are continued with g = {y, z, z^2, z^3, xy, xz, xz^2, xz^3, yz, yz^2, yz^3, xyz, xyz^2, xyz^3}. The conjugacy classes of H_3 are listed in the following:

(i) cl(1) = {1},
(ii) cl(x) = {x, xz} = cl(xz),
(iii) cl(x^2) = {x^2},
(iv) cl(xy) = {xy, x^3z} = cl(x^3z),
(v) cl(y) = {y, yz} = cl(yz),

(vi) cl(z) = {z},
(vii) cl(xy) = {xy, xyz} = cl(xyz),
(viii) cl(x^2y) = {x^2y, x^2yz} = cl(x^2yz),
(ix) cl(x^3y) = {x^3y, x^3yz} = cl(x^3yz),
(x) cl(x^2z) = {x^2z}.

Hence, $K(H_3)$ = 10. ■

CONCLUSION

Based on the calculation in Theorem 2.1, 2.2 and 2.3, it can be seen that the number of conjugacy classes for all three-generator groups of order 16 are equal to 10. However, the proof for each group is different due to the different relations given in the group presentation.

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