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On a Subclass of Tilted Starlike Functions with Respect to Conjugate Points

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ABSTRACT

We define $S_c^*(\alpha,\delta,A,B)$ be the class of functions which are analytic and univalent in an open unit disc, $E=\{z:|z|<1\}$ of the form $f(z)=z+a_2z^2+a_3z^3+\cdots+a_nz^n+\cdots=z+\sum_{n=2}^{\infty}a_nz^n$ and normalized with $f(0)=0$ and $f'(0)-1=0$ and satisfy $\left(e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right)\frac{1}{t_{\alpha\delta}}\prec\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, $z\in E$ where $g(z)=\frac{f(z)+\overline{f(\bar{z})}}{2}$, $t_{\alpha\delta}=\cos\alpha-\delta$, $\cos\alpha-\delta>0$, $0\le\delta<1$ and $|\alpha|<\frac{\pi}{2}$. The aim of this paper is to obtain the upper and lower bounds of $\operatorname{Re}\frac{zf'(z)}{g(z)}$ and $\operatorname{Im}\frac{zf'(z)}{g(z)}$ for this class of functions.

Keywords: **univalent functions, starlike functions with respect to conjugate points, subordination principle, bounds of** $\operatorname{Re}\frac{zf'(z)}{g(z)}$ **and** $\operatorname{Im}\frac{zf'(z)}{g(z)}$

INTRODUCTION

Let H be the class of functions ω which are analytic and univalent in the unit disc, $E=\{z:|z|<1\}$ given by

$$\omega(z)=\sum_{n=1}^{\infty}t_nz^n \tag{1}$$

and satisfies the conditions $\omega(0)=0,\ |\omega(z)|<1,\ z\in E.$

Let $P(A,B)$ be the class of all functions p of the form

$$p(z)=1+p_1z+p_2z^2+\cdots+p_nz^n+\cdots=1+\sum_{n=1}^{\infty}p_nz^n \tag{2}$$

that is analytic in E and satisfying the condition

$$p(z) \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1$$

for $z \in E$. Then this function is called a Janowski function. Hence, by using the definition of subordination it can be written that $p \in P(A,B)$ if and only if

$$p(z) = \frac{1+A\omega(z)}{1+B\omega(z)}, -1 \le B < A \le 1, \omega \in H.$$

Let S be the class of functions f which are analytic and univalent in E and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{3}$$

and normalized with $f(0)=0$ and $f'(0)-1=0$.

Let two functions $F(z)$ and $G(z)$ be analytic in E. If there exists a functions $\omega \in H$ which is analytic in E with $\omega(0)=0$ and $|\omega(z)|<1$ such that $F(z)=G(\omega(z))$ for every $z \in E$, then we say that $F(z)$ is subordinate to $G(z)$ and it can be written as $F(z) \prec G(z)$. We also note that if $G(z)$ is univalent in E, then the subordination is equivalent to $F(0)=G(0)$ and $F(E) \subset G(E)$.

Moreover, we introduce $S_c^*(\alpha,\delta)$ as the class of functions f which are analytic and univalent in E and of the form (3) and normalized with $f(0)=0$ and $f'(0)-1=0$ and satisfy

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right) > \delta \tag{4}$$

where $g(z) = \dfrac{f(z)+\overline{f(\bar{z})}}{2}$, $\cos\alpha - \delta > 0$, $0 \le \delta < 1$ and $|\alpha| < \dfrac{\pi}{2}$. We shall first relate the class $P(A,B)$ with the class $S_c^*(\alpha,\delta,A,B)$ so that we are able to obtain the bounds of $\operatorname{Re}\dfrac{zf'(z)}{g(z)}$ and $\operatorname{Im}\dfrac{zf'(z)}{g(z)}$ for the class $S_c^*(\alpha,\delta,A,B)$.

Theorem 1.1

If $f \in S$. Then $f \in S_c^*(\alpha,\delta,A,B)$ if and only if

$$\left(e^{i\alpha}\frac{zf'(z)}{g(z)} - \delta - i\sin\alpha\right)\frac{1}{t_{\alpha\delta}} \in P(A,B) \tag{5}$$

where $g(z) = \dfrac{f(z)+\overline{f(\bar{z})}}{2}$ and $t_{\alpha\delta} = \cos\alpha - \delta$.

Proof.

Let $f \in S_c^*(\alpha,\delta,A,B)$. From the fact that $\dfrac{zf'(z)}{g(z)} = p(z)$ where $g(z) = \dfrac{f(z)+\overline{f(\bar{z})}}{2}$ and g is starlike (Ravichandran , 2004), it follows that

$$\frac{zf'(z)}{g(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n. \tag{6}$$

Thus, from (4) we have

$$e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta=e^{i\alpha}\left(1+\sum_{n=1}^{\infty}b_nz^n\right)-\delta,$$

$$e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta=(\cos\alpha+i\sin\alpha)+e^{i\alpha}\sum_{n=1}^{\infty}b_nz^n-\delta,$$

$$e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha=t_{\alpha\delta}+e^{i\alpha}\sum_{n=1}^{\infty}b_nz^n \tag{7}$$

where $t_{\alpha\delta}=\cos\alpha-\delta.$

Rearranging (7), we get

$$e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha=t_{\alpha\delta}\left(1+\frac{e^{i\alpha}}{t_{\alpha\delta}}\sum_{n=1}^{\infty}b_nz^n\right),$$

$$\left(e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right)\frac{1}{t_{\alpha\delta}}=1+\frac{e^{i\alpha}}{t_{\alpha\delta}}\sum_{n=1}^{\infty}b_nz^n.$$

Hence,

$$\left(e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right)\frac{1}{t_{\alpha\delta}}=1+\sum_{n=1}^{\infty}p_nz^n$$

where $p_n=\dfrac{e^{i\alpha}b_n}{t_{\alpha\delta}}.$

Thus, for any $f\in S$, let

$$\left(e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right)\frac{1}{t_{\alpha\delta}}=p(z),z\in E \tag{8}$$

so that $f\in S_c^*(\alpha,\delta,A,B)$ if and only if $p\in P(A,B).$

Remark 1.2: We note that $t_{\alpha\delta}=\cos\alpha-\delta$ must always be positive so that (8) is valid. Therefore, we have to consider the condition of $\cos\alpha>\delta$ in the definition of the class $S_c^*(\alpha,\delta,A,B).$

We now in the position to represent our class of functions in terms of subordination.

Definition 1.3

$f\in S_c^*(\alpha,\delta,A,B)$ if and only if

$$\left(e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right)\frac{1}{t_{\alpha\delta}}\prec\frac{1+Az}{1+Bz},z\in E.$$

(9)

By definition of subordination, it follows that $f\in S_c^*(\alpha,\delta,A,B)$ if and only if

$$\left(e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right)\frac{1}{t_{\alpha\delta}}=\frac{1+A\omega(z)}{1+B\omega(z)}=p(z),\omega\in H \tag{10}$$

The following lemma due to Dixit and Pal (1995) is required to prove the later results.

Lemma 1.4

Let p be analytic in E. Then,

$$p(z)\prec\frac{1+Az}{1+Bz},-1\le B<A\le1$$

if and only if

$$\left| p(z) - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{(A-B)r}{1-B^2r^2}, |z| = r. \tag{11}$$

Further, if p *satisfies the inequality (11), then for* $|z| = r < 1$

$$\frac{1-Ar}{1-Br} \le \operatorname{Re} p(z) \le \frac{1+Ar}{1+Br}.$$

MAIN RESULTS

Theorem 2.1

If $f \in S_c^*(\alpha, \delta, A, B)$, *then for* $|z| = r < 1$ *we have*

$$\left| \frac{zf'(z)}{g(z)} - \left(\frac{1-B^2r^2 - Br^2e^{-i\alpha}T}{1-B^2r^2} \right) \right| \le \frac{Tr}{1-B^2r^2} \tag{12}$$

which gives the centre, $c(r)$ *and radius,* $d(r)$ *for functions in the class* $S_c^*(\alpha, \delta, A, B)$ *as*

$c(r) = \dfrac{1-B^2r^2 - Br^2e^{-i\alpha}T}{1-B^2r^2}$ *and* $d(r) = \dfrac{Tr}{1-B^2r^2}$ *for which* $g(z) = \dfrac{f(z) + \overline{f(\bar{z})}}{2}$, $T = (A-B)t_{\alpha\delta}$ *and* $t_{\alpha\delta} = \cos\alpha - \delta$.

Proof.

Using (10), the transformation maps $|\omega(z)| \le r$ onto the circle

$$\left| p(z) - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{(A-B)r}{1-B^2r^2}, |z| = r \tag{13}$$

and also

$$\left(e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta - i\sin\alpha \right) \frac{1}{t_{\alpha\delta}} = p(z)$$

where $t_{\alpha\delta} = \cos\alpha - \delta$.

Thus from (13), we get

$$\left| \frac{1}{t_{\alpha\delta}} \left(e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta - i\sin\alpha \right) - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{(A-B)r}{1-B^2r^2}, |z| = r. \tag{14}$$

Then, rearranging (14), we obtain

$$\left| e^{i\alpha} \frac{zf'(z)}{g(z)} - \left\{ \frac{(i\sin\alpha + \delta)(1-B^2r^2) + (1-ABr^2)t_{\alpha\delta}}{1-B^2r^2} \right\} \right| \le \frac{Tr}{1-B^2r^2}$$

where $T = (A-B)t_{\alpha\delta}$ and $t_{\alpha\delta} = \cos\alpha - \delta$,

$$\left| e^{i\alpha} \frac{zf'(z)}{g(z)} - \left(\frac{i\sin\alpha - B^2r^2 i\sin\alpha + \delta - \delta B^2r^2 + t_{\alpha\delta} - ABr^2\cos\alpha + \delta ABr^2}{1-B^2r^2} \right) \right| \le \frac{Tr}{1-B^2r^2},$$

$$\left|e^{i\alpha}\frac{zf'(z)}{g(z)}-\left(\frac{e^{i\alpha}-B^2r^2(i\sin\alpha+\delta)-ABr^2t_{\alpha\delta}}{1-B^2r^2}\right)\right|\le\frac{Tr}{1-B^2r^2},$$

$$\left|e^{i\alpha}\frac{zf'(z)}{g(z)}-\left(\frac{e^{i\alpha}-B^2r^2(i\sin\alpha+\delta)-ABr^2t_{\alpha\delta}+B^2r^2t_{\alpha\delta}-B^2r^2t_{\alpha\delta}}{1-B^2r^2}\right)\right|\le\frac{Tr}{1-B^2r^2},$$

$$\left|e^{i\alpha}\right|\left|\frac{zf'(z)}{g(z)}-\left(\frac{1-B^2r^2-Br^2e^{-i\alpha}T}{1-B^2r^2}\right)\right|\le\frac{Tr}{1-B^2r^2}.$$

Since $\left|e^{i\alpha}\right|=1,$ we obtain

$$\left|\frac{zf'(z)}{g(z)}-\left(\frac{1-B^2r^2-Br^2e^{-i\alpha}T}{1-B^2r^2}\right)\right|\le\frac{Tr}{1-B^2r^2} \tag{15}$$

which yields the center, $c(r)$ and radius, $d(r)$ where

$$c(r)=\frac{1-B^2r^2-Br^2e^{-i\alpha}T}{1-B^2r^2}$$

and

$$d(r)=\frac{Tr}{1-B^2r^2}.$$

Remark 2.2: The result now follows from the subordination principle. From Lemma 1.4 and Theorem 2.1, it follows that,
Let p be analytic in E. Then

$$\left(e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right)\frac{1}{t_{\alpha\delta}}=\frac{1+A\omega(z)}{1+B\omega(z)}\prec\frac{1+Az}{1+Bz},-1\le B<A\le1$$

if and only if

$$\left|\frac{zf'(z)}{g(z)}-\left(\frac{1-B^2r^2-Br^2e^{-i\alpha}T}{1-B^2r^2}\right)\right|\le\frac{Tr}{1-B^2r^2}$$

where $T=(A-B)t_{\alpha\delta}$ and $t_{\alpha\delta}=\cos\alpha-\delta.$
Thus, we can conclude that the Definition 1.3 holds.

Theorem 2.1 enables us to determine the upper and lower bounds of $\operatorname{Re}\frac{zf'(z)}{g(z)}$ and $\operatorname{Im}\frac{zf'(z)}{g(z)}$ as in the following theorem.

Theorem 2.3
If $f\in S_c^*(\alpha,\delta,A,B)$, *then for* $|z|=r,\ 0<r<1$

$$\frac{1-Tr-Br^2(B+T\cos\alpha)}{1-B^2r^2}\le\operatorname{Re}\frac{zf'(z)}{g(z)}\le\frac{1+Tr-Br^2(B+T\cos\alpha)}{1-B^2r^2} \tag{16}$$

and

$$\frac{1-Tr-Br^2(B-T\sin\alpha)}{1-B^2r^2}\le\operatorname{Im}\frac{zf'(z)}{g(z)}\le\frac{1+Tr-Br^2(B-T\sin\alpha)}{1-B^2r^2} \tag{17}$$

for which $g(z)=\dfrac{f(z)+\overline{f(\bar{z})}}{2}$, $T=(A-B)t_{\alpha\delta}$ *and* $t_{\alpha\delta}=\cos\alpha-\delta.$

Proof.
From Theorem 2.1, we have

$$\left|\frac{zf'(z)}{g(z)}-\left(\frac{1-B^2r^2-Br^2e^{-i\alpha}T}{1-B^2r^2}\right)\right|\le\frac{Tr}{1-B^2r^2}$$

which implies

$$\frac{1-Tr-Br^2(B+T\cos\alpha)}{1-B^2r^2}\le\operatorname{Re}\frac{zf'(z)}{g(z)}\le\frac{1+Tr-Br^2(B+T\cos\alpha)}{1-B^2r^2}$$

and

$$\frac{1-Tr-Br^2(B-T\sin\alpha)}{1-B^2r^2}\le\operatorname{Im}\frac{zf'(z)}{g(z)}\le\frac{1+Tr-Br^2(B-T\sin\alpha)}{1-B^2r^2}.$$

This completes the proof.

Remark 2.4: By putting $A=1$ and $B=-1$ in Theorem 2.3, we obtain the result for the class $S_c^*(\alpha,\delta,1,-1)$ which is introduced earlier as in (4) where

$$\frac{1-2rt_{\alpha\delta}-r^2(1-2t_{\alpha\delta}\cos\alpha)}{1-r^2}\le\operatorname{Re}\frac{zf'(z)}{g(z)}\le\frac{1+2rt_{\alpha\delta}-r^2(1-2t_{\alpha\delta}\cos\alpha)}{1-r^2}$$

and

$$\frac{1-2rt_{\alpha\delta}-r^2(1+2t_{\alpha\delta}\sin\alpha)}{1-r^2}\le\operatorname{Im}\frac{zf'(z)}{g(z)}\le\frac{1+2rt_{\alpha\delta}-r^2(1+2t_{\alpha\delta}\sin\alpha)}{1-r^2}.$$

The results obtained can also be reduced to the results for some subclasses such as $S_c^*(0,0,1,-1)$, $S_c^*(0,\delta,1,-1)$ and $S_c^*(0,0,A,B)$ which are introduced by El-Ashwah and Thomas (1987), Abdul Halim (1991) and Mad Dahhar and Janteng (2009) respectively.

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