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Some Bounds of Solution of First Order Dual Equations Under Uncertainty

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ABSTRACT

In this paper, we approximately deal with the solutions of first order dual equations under uncertainty. Indeed, we derive some bounds for lower and upper solutions obtained through the equality of width and cores of both sides of original fuzzy problems. By considering mentioned conditions together with fuzzy number conditions we extract some new and interesting approaches to determine some intervals included the support of solution of fuzzy equation. For sake of simplicity and considering the problem in real cases, we assume that the right-hand side of original problem is triangular.
Moreover, we have solved some new economic examples with fuzzy right-hand side in order to show the effect of fuzzy framework on the bound of solutions.
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Keywords: First order fuzzy equations in dual form, Width, Core, Fuzzy number.

INTRODUCTION

Linear systems of equations, with uncertainty on the parameters, play a major role in several applications in various areas such as economics, finance, engineering and physics. Hence, it is important to develop mathematical models and numerical procedures that would appropriately treat fuzzy linear systems and solve them.

The $n\times n$ fuzzy linear system has been studied by several authors. Buckley and Qu in (1991) construct solutions to the fuzzy matrix equation $\tilde{A}\tilde{x}=\tilde{b}$ when the elements in $\tilde{A}$ and $\tilde{b}$ re triangular fuzzy numbers. They presented six solutions of which they shown that five are identical. Ma et al. (2000), starting from the work by Friedman et al (1998) analyze the duality of fuzzy systems. They remark that the system $A_1x = A_2x + b$ is not equivalent to the system $(A_1 - A_2)x = b$, since for an arbitrary fuzzy number *u* there exists no element *v* such that *u+v=0.*

Consequently, Muzzilio and Reynaerts (2006) proposed a generalization of the vector solution of Buckley and Qu (1991) to $A_1x + b_1 = A_2x + b_2$: they gave the conditions under which the system has a vector solution and they showed that the linear systems $\tilde{A}\tilde{x}=\tilde{b}$ and $\tilde{A}_1\tilde{x}+\tilde{b}_1=\tilde{A}_2\tilde{x}+\tilde{b}_2$, with $\tilde{A}=\tilde{A}_1-\tilde{A}_2$ and $\tilde{b}=\tilde{b}_2-\tilde{b}_1$ have the same vector solutions, where '$-$' denotes the standard difference

In computational methods, several authors presented the approximate method for solving fuzzy linear and non-linear systems that some of which are for the dual form (Allahviranloo, and Salahshour (2011a), Allahviranloo, and Salahshour (2011b), Allahviranloo et al. (2011), Tavassoli Kajani et al. (2005), Salahshour and Homayoun nejad (2013)) .

The aim of this research is to present a novel approach for finding some reasonable bounds of the dual form of fuzz linear equation. The rest of the paper is organized as follows: In Section 2 the some required basic definitions of fuzzy settings theory is recalled. Afterwards, Section 3 has been devoted to the proposed method. In Section 4 to present the validation of the method, two examples are solved with details. A few conclusions are given in Section 5.

BASIC DEFINITIONS

In this section the definitions related to fuzzy number and its properties are reviewed which the interested readers can refer to Zimmermann (1985).

Definition 2.1. Let $u: X \to [0,1]$ be a fuzzy set. The **level sets** of A are defined as the classical sets
$A_\alpha = \{x \in X \mid A(x) \geq \alpha\}$,
$0 < \alpha \leq 1$.
$A_1 = \{x \in X \mid A(x) \geq 1\}$,
is called the **core** of the fuzzy set A, while

supp A $= \{x \in X \mid A(x) \geq 0\}$,
is called the **support** of the fuzzy set.

Definition 2.2. Let us denote by $\Re_F$ the class of fuzzy subsets of the real axis satisfying the following properties:

i. u is normal, i.e., there exists $s_0 \in \Re_F$ such that $u(s_0) = 1$,

ii. u is a convex fuzzy set (i.e.) $u(\lambda s + (1-\lambda)r) \geq \min\{u(s), u(r)\}$, $\forall \lambda \in [0,1], s, r \in \Re.$

iii. u is upper semicontinuous on $\Re_F$.

iv. $cl\{s \in \Re \mid u(s) \succ 0\}$ is compact where *cl* denotes the closure of a subset.

Then $\Re_F$ is called the space of fuzzy numbers. Clearly $\Re \subset \Re_F$. $[u]^\alpha = \{s \in \Re \mid u(s) \geq \alpha\}$ be denoted for $0 < \alpha \leq 1$ and $[u]^0 = \{s \in \Re \mid u(s) > 0\}$. The notation $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ denotes the α-level set of u.

Definition 2.3. Consider the following $n \times n$ linear system of equations

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \ldots + \tilde{a}_{1n}\tilde{x}_n + \tilde{b}_1 = \tilde{c}_{11}\tilde{x}_1 + \tilde{c}_{12}\tilde{x}_2 + \ldots + \tilde{c}_{1n}\tilde{x}_n + \tilde{d}_1 \\ \tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \ldots + \tilde{a}_{2n}\tilde{x}_n + \tilde{b}_2 = \tilde{c}_{21}\tilde{x}_1 + \tilde{c}_{22}\tilde{x}_2 + \ldots + \tilde{c}_{2n}\tilde{x}_n + \tilde{d}_2 \\ \vdots \\ \tilde{a}_{n1}\tilde{x}_1 + \tilde{a}_{n2}\tilde{x}_2 + \ldots + \tilde{a}_{nn}\tilde{x}_n + \tilde{b}_n = \tilde{c}_{n1}\tilde{x}_1 + \tilde{c}_{n2}\tilde{x}_2 + \ldots + \tilde{c}_{nn}x_n + \tilde{d}_n \end{cases}$$

where the elements $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij}$ and $\tilde{d}_i, 1 < i, j < n$ are fuzzy numbers is so-called **fuzzy linear systems in dual form (DFFLS).**

THE SOLUTION METHOD

Let us consider the following fuzzy linear equation in the dual form:

$$a\tilde{x} = b\tilde{x} + \tilde{d} \qquad (3.1)$$

where a, b are real numbers and $\tilde{d}$ is a triangular fuzzy number. The main aim is to obtain some bounds for the solution of Eq. (3.1).
Before proceed, we should mention that if Eq. (3.1) has exact solution, the each property of left-hand side of (3.1) is equal to the same property for the right-hand side of (3.1). In other words, width of $a\tilde{x}$ must be equal to core of $b\tilde{x} + \tilde{d}$, too.

In a real position, obtaining the exact solution is not valid. In fact, because of existing uncertainty in the real-type model, it is better (and it is coincided with real manner) to obtain some reasonable bounds for the solution together with the following essential properties.

(A) The solution in 1-cut position belongs to this interval solution.
(B) Obtaining the lower and upper bounds of interval with a systematic approach without using measure.
(C) The support of exact solution is also belongs to the proposed interval solution (as a bound of solution).

Remark. Notice that in the fuzzy framework, the definition of bound for the solution of Eq. (1), may be have some different interpretations. Here, we consider this definition such that the support of exact solution is included in the interval solution, i.e., if $\operatorname{supp} x_{exact}[\underline{x}, \overline{x}]$ and the proposed interval solution is $[x^l, x^r]$, then

$$\operatorname{supp} x_{exact} \subseteq [x^l, x^r].$$

Now, we are at position to describe the method. For the sake of simplicity, we assume that a and b are positive real numbers and $\tilde{d}$ in a parametric form (α -cut) denoted by $\tilde{d}(\alpha) = [d^l(\alpha), d^r(\alpha)]$. If the original problem (3.1) has an exact solution, we have

$$\begin{cases} w(a\tilde{x}) = w(b\tilde{x} + \tilde{d}), \\ Core(a\tilde{x}) = Core(b\tilde{x} + \tilde{d}), \end{cases} \qquad (3.2)$$

where $w(.)$ denotes the width, and it is calculated by $width(\tilde{x}) = x^r - x^l$,

where x^r and x^l are upper and lower points of support, respectively.
Using relation (3.2), we have

$$\begin{cases} a(x^r - x^l) = b(x^r - x^l) + (d^r - d^l) \\ ax^c = bx^c + d^c. \end{cases}, \quad (3.3)$$

Now, we seek some unknown values x^l and x^r satisfy in the following condition:
$x^l \le x^c \le x^r$,

or equivalently, $(a-b)x^l \le d^c \le (a-b)x^r$ with assumption $a > b$.
Set

$$(a-b)x^l \le d^c \le (a-b)x^r \to$$
$$(a-b)(x^r - x^l) = d^r - d^l \to$$
$$(a-b)x^r - (a-b)x^l = d^r - d^l \ge d^c - (a-b)x^l \to$$
$$d^r - d^l - d^c \ge -(a-b)x^l \to$$
$$-d^r + d^l + d^c \le (a-b)x^l \to x^l \ge \frac{-d^r + d^l + d^c}{a-b},$$

and using the fact that $(b-a)x^r \le -d^c \le (b-a)x^l$, we have:

$$(a-b)x^r - (a-b)x^r \le (a-b)x^r - d^c \le (a-b)x^r - (a-b)x^l = d^r - d^l$$
$$\to (a-b)x^r \le d^r - d^l + d^c \to x^r \le \frac{d^r - d^l + d^c}{a-b}$$

Remark. As we stated previously, by considering the essential properties (A) and (B), we find that
$x^l \le x^c \le x^r$
That is interesting result without solving optimization problem. In fact, the support of solution is as $[\underline{x}(.), \overline{x}(.)] \subseteq [x^l, x^r]$. In other words, we obtain some reasonable bounds for the solution of (3.1) without using common ways.

EAMPLES

In this section two examples are considered to depict the validation and efficiency of the presented method for the solution of DFLS.

Example 4.1. Consider the following fuzzy first order dual equation
$4\tilde{x} = 3\tilde{x} + \tilde{d}$, (4.1)

Where $\tilde{d}(\alpha) = [1+\alpha, 3-\alpha], \alpha \in [0,1].$

It is easy to verify that $\tilde{x}_{exact}(\alpha) = [1+\alpha, 3-\alpha].$ Now, we obtain some reasonable bounds for the solution of Eq. (3.1). Using the lower bound for x^l and upper bound for x^r, we have:

$$x^l \ge \frac{-d^r + d^l + d^c}{a-b} = 0 \quad \text{and} \quad x^r \ge \frac{d^r - d^l + d^c}{a-b} = 4.$$

So, we have supp $\tilde{x}=[1,3]\subseteq[0,4].$ Also, one can easily obtain that $x^c=2\in[0,4].$
Moreover, if we investigate the fuzzy bound, instead of interval bound for support, we can construct such fuzzy bound, denoted by $\tilde{M}$ which is built as follows:

$$\begin{cases}\text{supp } \tilde{M}=[0,4],\\ Core\ \tilde{M}=2,\end{cases}$$

then, we have $\tilde{M}(\alpha)=[2\alpha,4-2\alpha].$ In fact, $\tilde{x}_{exact}(\alpha)\subseteq\tilde{M}(\alpha)$.
In other words, we can find that $[1+\alpha,3-\alpha]\subseteq[2\alpha,4-2\alpha]$ for all $\alpha\in[0,1]$. Since, $1+\alpha\geq2\alpha$ and $3-\alpha\geq4-2\alpha$ for each $\alpha\in[0,1]$.

Example 4.2. Consider the following fuzzy dual equation:

$$8\tilde{x}=5\tilde{x}+\tilde{d}, \quad (4.2)$$

where $\tilde{d}(\alpha)=[\alpha^2,2-\alpha^2],\alpha\in[0,1]$. The exact solution is $\tilde{x}_{exact}(\alpha)=[\frac{\alpha^2}{3},\frac{2-\alpha^2}{3}]$. Also, the lower bound for x^l and upper bound for x^r are obtained as follows:

$$\begin{cases}x^l\geq-\frac{1}{3},\\ x^r\leq1.\end{cases}$$

So, the interval bound for the solution is $[-\frac{1}{3},1]$.
Moreover, we have the following results:

$$\begin{cases}\text{supp } \tilde{M}=[-\frac{1}{3},1]\\ Core\ \tilde{M}=\frac{1}{3}.\end{cases}$$

Therefore, $\tilde{M}(\alpha)=[-\frac{1}{3}+\frac{2}{3}\alpha^2,1-\frac{2}{3}\alpha^2]$. It is easy to verify that $\tilde{x}_{exact}(\alpha)\subseteq\tilde{M}(\alpha)$, for each $\alpha\in[0,1]$. Since,

$$\frac{\alpha^2}{3}\geq-\frac{1}{3}+\frac{2}{3}\alpha^2 \quad \text{and} \quad \frac{2-a^2}{3}\leq1-\frac{2}{3}\alpha^2.$$

Conclusion

In this paper, we have proposed a novel approach to obtain some reasonable bounds, both intervals bounds for support and fuzzy bound for the solution in each α-cut. Some illustrative examples were given to depict this new approach.

REFERENCES

Buckley, J.J., Qu. Y., (1991) , Solving system of linear fuzzy equations, *Fuzzy Sets and Systems* **43**: 33–43.

Friedman, M., Ming, Ma., Kandel, A., (1998) , Fuzzy linear system, *Fuzzy Sets and Systems* **96**: 209–261.

Ming, Ma., Friedman, M., Kandel, (2000), A., Duality in fuzzy linear systems, *Fuzzy Sets and Systems* **109:** 55–58.

Muzzilio, S., Reynaerts, H., (2006) , Fuzzy linear system of the form $A_1x + b_1 = A_2x + b_2$, *Fuzzy Sets and Systems* **157:** 939–951

Allahviranloo, T., Salahshour, S., (2011a), Fuzzy symmetric solutions of fuzzy linear systems, *Journal of Computational and Applied Mathematics* **235**: 4545–4553.

Allahviranloo, T., Salahshour, S., (2011b) Bounded and symmetric solutions of fully fuzzy linear systems in dual form, *Procedia Computer Science* **3:** 1494–1498.

Allahviranloo, T., Salahshour, S., Khezerloo, M., (2011) Maximal- and minimal symmetric solutions of fully fuzzy linear systems, *Journal of Computational and Applied Mathematic*s **235**: 4652–4662.

Tavassoli Kajani, M., Asady, B., Hadi Vencheh, A., (2005), An iterative method for solving dual fuzzy nonlinear equations, *Applied Mathematics and Computation* **167**: 316–323.

Salahshour, S., Homayoun nejad, M., (2013), Approximating solution of fully fuzzy linear systems in dual form, *Int. J. Industrial Mathematics*, **Vol. 5, Article ID IJIM-00223**, 5 pages.

Zimmermann, H.J. , (1985), Fuzzy Sets Theory and Applications, Kluwer, Dorrecht,.