Menemui Matematik (Discovering Mathematics)
Vol. 35, No. 2: 38 – 45 (2013)

Phase-fitted Explicit Runge-Kutta Method for Periodic Initial Value Problems

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ABSTRACT

In this paper phase-fitted explicit Runge-Kutta method is derived for solving first-order ordinary differential equations with periodical solutions. The phase-lag and dissipation properties for Runge-Kutta (RK) methods is also discussed. The new method has algebraic order three with phase-lag of order infinity. The numerical results for the new method is compared with existing method when solving the first-order differential equations with periodic solutions using constant step size.

Keywords: **Runge-Kutta methods; Phase-lag; Oscillatory solutions**

INTRODUCTION

In this paper, we are focused on initial value problems (IVP) related to first-order ODEs of the form:

$$y'(x) = f(x, y), \;\; y(x_0) = y_0, \qquad (1)$$
$$y'(x_0) = y'_0 \qquad x \in [a, b]$$

where

$$y(x) = [y_1(x), y_2(x), \dots, y_s(x)]^T$$
$$f(x, y) = [f_1(x, y), f_2(x, y), \dots, f_s(x, y)]^T$$

and y_0 is a given vector of initial condition and their solution is oscillating. Such problems often arise in different fields of engineering and applied sciences such as structural mechanics, astronomy, physical chemistry, chemical physics and electronics.

There are many procedures in order to develop efficient methods for the numerical solution of the equation (1) such as phase fitting, P-stability, and methods with minimal phase-lag. The results of these procedures are multistep methods (linear multistep (two-step or four-step) and hybrid multistep methods) (see Cash et al. (1990), Simos et al. (1990), Avdelas and Simos (1996a,1996b), Simos (1999), Ahmad et al. (2013a, 2013b), Senu et al. (2010a, 2010b, 2010c, 2011). From the above remark it is obvious that there is no efficient one-step method for the numerical solution of the equation (1). This is important since for the numerical solution of any problem using an one-step method, only the initial condition is required, while for the numerical solution of the same problem using a multistep method many initial conditions can be required. The first of them is the condition given by the problem. The rest are conditions that can produce errors that are much greater than the error of the numerical method. For this reason, it is important to investigate the production of effcient one-step methods and especially the well-known Runge-Kutta methods.

In 1980, the term phase-lag was first introduced by Brusa and Nigro (1980). For the past three decades, several authors have developed RKN methods based on the minimal phase-lag theory. See Van der Houwen and Sommeijer (1987), Senu et al. (2009), and Van de Vyver (2007). In 1993, Simos derived a Runge-Kutta-Fehlberg method based on the idea of phase-lag of order infinity (1993).

In this paper, we will derive a new explicit RK method with three-stage third-order with dispersion of order infinity. In the next sections, we will discuss some basic theory of Runge-Kutta methods. Then, the construction of the new phase-fitted Runge-Kutta method is described and its coefficients are displayed in the Butcher table. Finally, numerical tests are performed on first-order differential equation problems which are known have oscillatory solutions.

GENERAL THEORY

RK methods for the numerical integration of the Initial Value Problem (IVP) is given by

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f_i \tag{2}$$

where

$$f_1 = f(x_n, y_n)$$
$$f_i = f\left(x_n + c_i h\,, y_n + h\sum_{j=1}^{i-1} a_{ij} f_j\right), i = 1\,, \dots, s$$

The parameters a_{ij}, b_{i} and c_{i} are assumed to be real and $j \geq i$ then $a_{i,j} = 0$. m is the number of stages of the method. All the parameter can be tabulate in Butcher Tableau in the following form

C	A
	b^T

where

$$C = [c_1, c_2, \dots, c_m]^T, A = \left[a_{i,j}\right], b^T = [b_1, b_2, \dots, b_m]^T.$$

Consider the standard test problem of differential equation

$$y' = f(x.y) = \lambda y, \qquad \text{and} \qquad y(x_n) = y_n \tag{3}$$

which has true solution

$$y(x) = y_n e^{\lambda(x - x_n)}.$$

Applying the test equation (3) to the RK formula (2), by setting $v = \lambda h$ and factoring we obtain

$$y_{n+1} = R(v) y_n, \qquad \text{where} \quad |R(v)| < 1, \tag{4}$$
$$y_{n+1} = \{1 + v b_i (I - vA)^{-1}\} y_n. \tag{5}$$

$R(v) = \breve{a}(v)$ is said to be stability polynomial of RK method.

Definition 2.1: The quantities of the stability equation (5) corresponding to RK methods (2)

$$\phi(v) = v - \arg[\breve{a}], \tag{6}$$
$$\alpha(v) = 1 - |\breve{a}(v)| \tag{7}$$

are called the dispersion (or phase lag or phase error) and the amplification error respectively. If $\phi(v) = O(v^{q+1})$ and $\alpha(v) = O(v^{r+1})$ then the method is said to be dispersive of order q and dissipative of order r.

From Houwen and Sommeijer (1987), equations (6) and (7) are given by

$$\phi(v) = v - \arctan\left(v\frac{B_m(v^2)}{A_m(v^2)}\right), \tag{8}$$

$$\alpha(v) = 1 - \sqrt{A_m^2(v^2) + v^2 B_m^2(v^2)} \tag{9}$$

for dispersion and dissipation respectively and function $R(v)$ can be written as

$$R(v) = A_m(v^2) + ivB_m(v^2), \qquad v = \lambda h \tag{10}$$

where

$$A_m(z) = 1 - \beta_2 z + \beta_4 z^2 + \cdots,$$
$$B_m(z) = 1 - \beta_3 z + \beta_5 z^2 + \cdots,$$

for $j > m$, $\beta_j = 0$ and $z = v^2$.

CONSTRUCTION OF THE NEW METHOD

In this section, we will derive a three-stage third order explicit RK method with phase-lag of order infinity. The derivation of the new RK method is based on the method derived by Dormand (1996) as given in Table 1

Table 1: Butcher Tableau for third-order RK method

0	0		
$\frac{1}{2}$	$\frac{1}{2}$	0	
$\frac{3}{4}$	0	$\frac{3}{4}$	0
	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$

Consider the absolute stability for three-stage RK method and letting a_{31} as free parameter, we have :

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad A=\begin{bmatrix}0&0&0\\ \frac{1}{2}&0&0\\ a_{31}&\frac{3}{4}&0\end{bmatrix},\quad e=\begin{bmatrix}1\\1\\1\end{bmatrix}, b=\begin{bmatrix}\frac{2}{9}&\frac{1}{3}&\frac{4}{9}\end{bmatrix}.$$

As it has already been defined in (8), in order to have phase-lag of order infinity, the following equation must hold. That is

$$v-\arctan\left(v\frac{B_m(v^2)}{A_m(v^2)}\right)=0.$$

By applying $A_m(v^2)$ and $B_m(v^2)$ to the above formula and solving for a_{31}, then we get

$$a_{31}=\frac{3(6\tan(h)-3\tan(h)\,h^2+h^3-6h)}{8(\tan(h)\,h^2}.$$

The expansion Taylor series for a_{31}, which is given from the above formula is:

$$a_{31}=-\frac{3}{40}h^2-\frac{1}{280}h^4-\frac{1}{3150}h^6-\frac{13}{415800}h^8-\frac{893}{283783500}h^{10}-\frac{271}{851350500}h^{12}+O(h^{13}).$$

The free parameter a_{31} is chosen in this method because it gives the minimum global error than the other coefficients. This new method is denoted as RK3P.

PROBLEMS TESTED AND NUMERICAL RESULTS

In this section, we will apply the new method to some differential equation problems. The following explicit RK methods are selected for the numerical comparisons:

- RK3P: The new derived third order RK method with phase-lag of order infinity.
- RK3: The three stage third order RK method derived by Dormand (1996).

The accuracy criteria taken is calculating the maximum error, Emax(X)

$$E\max(X)=\max\|y(x_n)-y_n\|,$$

where $x_n=x_0+nh, n=1,2,\ldots,\frac{X-x_0}{h}$.

Problem 1 :

$y'' = -64y,\; y(x_0) = 1, y'(x_0) = -2$

Theoretical solution : $y(t) = -\frac{1}{4}\sin(8t) + \cos(8t)$

Reduce to first order system :

$y' = y_2, y' = -64y_1$

Source : Van der Houwen and Someijer(1987)

Problem 2 :

$y'' = -100y + 99\sin(t),\qquad y(x_0) = 1,\qquad y'(x_0) = 11$

Theoretical solution : $y(t) = \cos(10t) + \sin(10t) + \sin(t)$

Reduce to first order system :

$y' = y_2, y' = -100y_1 + 99\sin(t)$

Source : Papageorgiou, Ch. Tsitouras and Papakostas (1993)

Problem 3:

$y'' = -y + 0.001\cos(t)\qquad y(0) = 1\qquad y'(0) = 0$

$y'' = -y + 0.001\sin(t)\qquad y(0) = 0\qquad y'(0) = 0.9995$

Theoretical solutions : $y(t) = \cos(t) + 0.0005t\sin(t)$

: $y(t) = \sin(t) - 0.0005t\cos(t)$

Reduce to first order system :

$y' = y_2, y' = -y_1 + 0.001\cos(t)$

$y' = y_4, y' = -y_3 + 0.001\sin(t)$

Source : Stiefel and Bettis (1969)

Table 2 : Comparison Maximum Global Error for RK3 and RK3P for Problem 1

h	Methods	End of Integration, *b*		
		100	1000	10000
0.003125	RK3	4.289762(-3)	4.283437(-2)	4.184669(-1)
	RK3P	8.582208(-4)	8.585832(-3)	8.545527(-2)
0.00625	RK3	3.425218(-2)	3.365043(-1)	2.810000(0)
	RK3P	6.865104(-3)	6.844086(-2)	6.595225(-1)
0.0125	RK3	2.699934(-1)	2.338203(0)	7.959376(0)
	RK3P	5.481962(-2)	5.324316(-1)	4.016349(0)
0.025	RK3	1.930219(0)	7.705566(0)	8.246237(0)
	RK3P	4.284972(-1)	3.422607(0)	8.207491(0)

Table 3: Comparison Maximum Global Error for RK3 and RK3P for Problem 2

h	Methods	End of Integration, *b*		
		100	1000	10000
0.003125	RK3	1.793812(-2)	1.786607(-1)	1.688584(0)
	RK3P	3.590799(-3)	3.592129(-2)	3.551654(-1)
0.00625	RK3	1.428522(-1)	1.367605(0)	9.031675(0)
	RK3P	2.871287(-2)	2.848369(-1)	2.604715(0)
0.0125	RK3	1.102794(0)	7.881169(0)	1.413917(+1)
	RK3P	2.284565(-1)	2.127641(0)	1.137469(+1)
0.025	RK3	6.774668(0)	1.413507(+1)	1.414262(+1)
	RK3P	1.738494(0)	1.033797(+1)	1.414218(+1)
0.05	RK3	1.419314(+1)	1.419314(+1)	1.419314(+1)
	RK3P	9.291349(0)	1.414268(+1)	1.414323(+1)

Table 4 : Comparison Maximum Global Error for RK3 and RK3P for Problem 3

h	Methods	End of Integration, *b*		
		100	1000	10000
0.003125	RK3	1.257515(-7)	1.307151(-6)	3.394642(-5)
	RK3P	2.509523(-8)	2.601839(-7)	6.618617(-6)
0.00625	RK3	1.006036(-6)	1.046492(-5)	2.739540(-4)
	RK3P	2.007616(-7)	2.088869(-6)	5.490743(-5)
0.0125	RK3	8.048552(-6)	8.371482(-5)	2.189958(-3)
	RK3P	1.606103(-6)	1.670983(-5)	4.380343(-4)
0.025	RK3	6.439034(-5)	6.695222(-4)	1.748343(-2)
	RK3P	1.284961(-5)	1.336775(-4)	3.502254(-3)
0.05	RK3	5.150657(-4)	5.343898(-3)	1.376125(-1)
	RK3P	1.028197(-4)	1.068936(-3)	2.793450(-2)

From Table 2-4, we can see that the RK3P method is always more accurate than the RK3 method.

CONCLUSION

In this paper, we have derived a new third order phase-fitted RK method. The new method is based on Dormand's third algebraic order RK method. Numerical results show that the new method is more accurate for solving first-order differential equations with oscillating solutions.

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