

Numerical Solutions of Biomechanical Model of a Cyclist using Runge-Kutta Methods

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ABSTRACT

Some biomechanical models are represented by nonlinear first order ordinary differential equations. The objective of this study is to determine the velocity of a biomechanical model that involves a cyclist coasting downhill. Two methods namely; the modified explicit and diagonally implicit fifth-order Runge-Kutta methods are utilised. The performance of the two methods is compared with the exact solution. Absolute errors obtained show that the implicit method gives better accuracy as compared with the explicit one.

Keywords: Modified explicit Runge-Kutta; modified diagonally implicit Runge-Kutta; nonlinear first order ordinary differential equations; biomechanical model

INTRODUCTION

Mathematical modelling is frequently used in many areas such as in science, engineering, medicine, economics and social sciences. Ordinary and partial differential equations are one of the important and widely used representations in various physical problems such as the motion of the planet in a gravity field like the Kepler problem, the simple pendulum, geometry, mechanics, astronomy, orthodontics, population, biomechanics, electrical circuits, biomedical system, weather prediction and chemical kinetics problems (Noorhelyna & Rokiah Rozita 2008). However, not all the differential equations can be solved analytically and even if it could be done, it is usually extremely difficult to perform. Thus, numerical methods are truly the fundamental component for solving differential equations that should not be neglected even though they only produce approximate solutions.

Mathematicians, in particular the numerical analysts, have continuously developed numerical methods for solving the differential equations since the late eighteenth century. Numerical methods have made enormous progress alongside the rapid development of computers. The earliest numerical method for solving ordinary differential equations is the famous Euler method which evaluates the driving function once in each step and uses an approximation solution from the previous step to update a solution (Din et al. 2007). The early extensions of this method are the well-known and most commonly used method; the Runge-Kutta methods, which comprise the second-order, third-order and fourth-order Runge-Kutta methods. Developed from Euler's method, Runge-Kutta methods are able to achieve higher order without sacrificing the one-step form. Runge-Kutta methods use a result given at the end of the previous step while evaluating functions at

the one or more off-step points. These methods can be used to solve complicated problems such as the nonlinear differential equations which usually do not produce analytical solution.

In the early days of Runge–Kutta methods, the aim is focused on finding explicit methods of higher and higher order (Butcher 1987). Many attempts have been made by numerical analysts to modify the classical Runge–Kutta method which involves the utilisation of certain means such as arithmetic, harmonic, contra harmonic and geometric mean either in the main equation or in the stages. These modified methods have been successfully developed by Ahmad and Yaacob (2005), Ahmad et al. (2008), Evans and Yaakub (1993a, 1993b), Sanugi and Evans (1993) and Wazwaz (1994). Traditionally, Runge–Kutta methods are all explicit however Butcher (1987) has listed some basic reasons for taking a serious interest in implicit Runge–Kutta methods. The main reason is because of the higher orders of accuracy in the implicit methods are better than the explicit ones. However, the main problem with all the implicit Runge–Kutta formulae, which have been proposed by Butcher, Ehle and Chipman (Alexander 1977) is that the solution of the resulting nonlinear equations is prohibitively expensive. Due to the excessive cost in evaluating the stages in a fully implicit Runge–Kutta method, many researchers have opted for the diagonally implicit Runge–Kutta (DIRK) method, as named by Alexander (1977).

Several numerical methods have been used to solve the biomechanical models (Ahmad et al. 2008; Che Nan & Rambely 2011). Among the models involved is cycling, badminton smash and load carriage problems. Cycling is a sport activity which provides a lot of benefit to a cyclist such as to maintain a healthy life style, stabilize heart beating and decrease the risk of getting cardiovascular illness.

BIOMECHANICAL MODEL

This article focuses on instituting the velocity of a biomechanical model for a cyclist coasting downhill where all the components on the surface are movement and force components i.e., gravitational force, normal force and air force. In order to establish the velocity of the model, it is assumed no friction acting in the system. Figure 1 shows the free body diagram of the model.

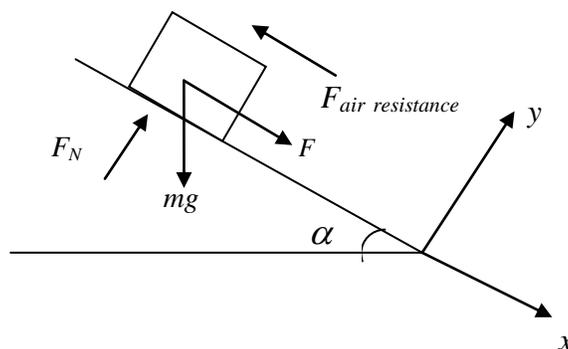


Figure 1. A body diagram of a cyclist coasting downhill

where F represents force, F_N represents normal force and $F_{air\ resistance}$ is the force for air resistance. This model is represented by a nonlinear first order ordinary differential equation

$$v' = g \sin \alpha - \frac{k}{m} v^2,$$

where g is the gravitational acceleration, α is the angle of the hill from horizontal line, m is total mass of the bicycle and cyclist, k is the constant value of air resistance and v is the velocity of the cyclist coasting downhill.

The exact solution for the above equation is given by

$$v = \sqrt{\frac{mg \sin \alpha}{k}} \tanh \left(\sqrt{g \sin \alpha} \frac{\sqrt{k}}{\sqrt{m}} t \right).$$

MODIFIED EXPLICIT AND IMPLICIT RUNGE-KUTTA METHODS

Two types of numerical methods, namely, the modified explicit and diagonally implicit fifth-order Runge-Kutta methods are applied to the biomechanical system in order to get the approximate numerical solution. The explicit fifth-order Runge-Kutta method is a modified method which was developed using the arithmetic means by Noorhelyna and Rokiah (2008). The modified implicit fifth-order Runge-Kutta method is due to Din et al. (2007) where it can be executed simultaneously using two processors. This method is a modification to the existing diagonally implicit Runge-Kutta (DIRK) method of Alexander (1977).

The modification of the explicit fifth-order Runge-Kutta method was done by substituting the arithmetic mean in the stages. Using *Mathematica*, a modified explicit fifth-order Runge-Kutta method, abbreviated as RK5(1), is constructed based on arithmetic mean and represented by the followings:

$$y_{n+1} = y_n + h(-3.7783286500685627k_1 + 0.18312885616492072k_2 + 0.04837565197099888k_3 + 17.700904612988186k_4 + 22.61398646725067k_5) \quad \text{where}$$

$$k_1 = f(x_n, y_n),$$

$$k_2 = f(x_n + 0.6826487126671337h, y + 0.6826487126671337hk_1),$$

$$k_3 = f\left(x_n + 2.7638749083367884h, y + 2.7638749083367884h \frac{k_1 + k_2}{2}\right),$$

$$k_4 = f\left(x_n + 0.1h, y + 0.1h \frac{k_1 + k_2 + k_3}{3}\right),$$

$$k_5 = f\left(x_n + 0.1h, y + 0.1h \frac{k_1 + k_2 + k_3 + k_4}{4}\right).$$

The formula for the modified implicit DIRK method, abbreviated as DIRK5, is represented using Butcher's array:

γ	γ					
β	0	β				
c_3	a_{31}	a_{32}	γ			
c_4	a_{41}	a_{42}	0	β		
c_5	a_{51}	a_{52}	a_{53}	a_{54}	γ	
c_6	a_{61}	a_{62}	a_{63}	a_{64}	0	β
	b_1	b_2	b_3	b_4	b_5	b_6

where the coefficients are (Din et al. 2007):

$a_{22} = \beta,$	$\gamma = 0.5111360444707436,$
$a_{32} = 0.3304320730047836,$	$\beta = 0.1200913622711499,$
$a_{33} = \gamma,$	$c_1 = \gamma,$
$a_{42} = -0.001764317609504879,$	$c_2 = \beta,$
$a_{43} = 0.0,$	$c_3 = 0.5083725832456476,$
$a_{44} = \beta,$	$c_4 = 0.1048607614397692,$
$a_{52} = 0.09184637904240046,$	$c_5 = 1 - \gamma,$
$a_{53} = -0.2402905535606616,$	$c_6 = 1 - \beta,$
$a_{54} = 0.206131006694386,$	$b_1 = 0.0,$
$a_{55} = \gamma,$	$b_2 = 0.0,$
$a_{62} = 0.008530162133924786,$	$b_3 = -0.179679897773612,$
$a_{63} = 0.5217820185104087,$	$b_4 = 0.2614074344151084,$
$a_{64} = 0.251411354949371,$	$b_5 = 0.6241305207277307,$
$a_{65} = 0.0,$	$b_6 = 0.2941419426307729,$
$a_{66} = \beta.$	

The values of a_{i1} for $i = 1, 2, \dots, 6$ are obtained by following the relationship,

$$a_{i1} = c_i - \sum_{j=2}^i a_{ij}.$$

NUMERICAL RESULTS

The numerical experiment is conducted by choosing $g = 9.81 \text{ m s}^{-1}$, $m = 75 \text{ kg}$, $\alpha = 0.17453 \text{ rad}$, $k = 0.135 \text{ N m}^{-1}$ at altitude 1000 m within the time range 12 to 120 s. Numerical results is obtained for every ten steps using RK5(1) and DIRK5 methods. Table 1 and Figure 2 show the performance's comparison between RK5(1) and DIRK5. The step size used is $h = 2$. The absolute error is defined as

$$\left(|y_i - y(x_i)| \right); \quad t = 12, 24, \dots, 120$$

where y_i is the computed value and $y(x_i)$ is the true solution of the problems. It is found that the DIRK5 gives better accuracy as compared with RK5(1). From the results, as time increased, DIRK5 shows smaller absolute errors compared with RK5(1).

Table 2 shows the comparison between RK5(1) and DIRK5 in terms of absolute errors with different step sizes ranging from $h = 0.1$ to $h = 2$ at time $t = 120 \text{ s}$. The exact value of the velocity at that particular time is $30.762972279556 \text{ m s}^{-1}$. Several different step sizes had also been chosen to obtain numerical solutions and absolute errors of both fifth-order Runge-Kutta methods at time $t = 120 \text{ s}$ as shown in Table 2 and Figure 3. It shows that DIRK5 gives better agreement to the true solution. However, RK5(1) still gives a comparable results.

Table 1. Absolute errors for RK5(1) and DIRK5 with step size, $h = 2$

t / s	Exact solution	RK5(1)	DIRK5	Absolute errors	
				RK5(1)	DIRK5
12	17.883767780880	18.519366118925	17.883713498186	6.36×10^{-01}	5.43×10^{-05}
24	26.732994042538	26.942099178303	26.732668182778	2.09×10^{-01}	3.26×10^{-04}
36	29.642121973158	29.692300111552	29.641816162795	5.02×10^{-02}	3.06×10^{-04}
48	30.462272948081	30.472580350886	30.462116561204	1.03×10^{-02}	1.56×10^{-04}
60	30.683151014580	30.684970822736	30.683088442055	1.82×10^{-03}	6.26×10^{-05}
72	30.741895813862	30.742127088769	30.741873579750	2.31×10^{-04}	2.22×10^{-05}
84	30.757467378162	30.757461071072	30.757459988276	6.31×10^{-06}	7.39×10^{-06}
96	30.761591288089	30.761571497560	30.761588933459	1.98×10^{-05}	2.35×10^{-06}
108	30.762683190606	30.762673093965	30.762682461785	1.01×10^{-05}	7.29×10^{-07}
120	30.762972279556	30.762968304820	30.762972058681	3.97×10^{-06}	2.21×10^{-07}

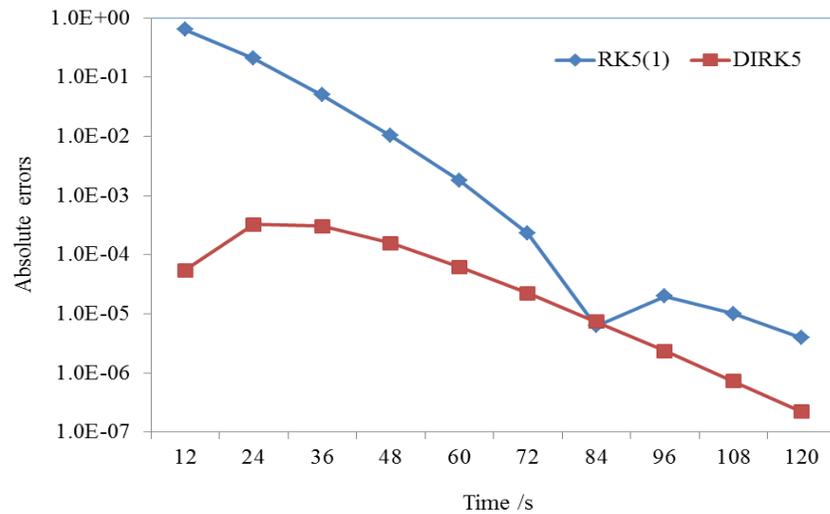


Figure 2. Absolute errors for the explicit vs implicit Runge-Kutta methods

Table 2. Absolute errors for RK5(1) and DIRK5 at time, $t = 120$ s

Step size, h	RK5(1)	DIRK5	Absolute errors	
			RK5(1)	DIRK5
2	30.762968304820	30.762972058681	3.97×10^{-06}	2.21×10^{-07}
1	30.762976551138	30.762972267785	4.27×10^{-06}	1.18×10^{-08}
0.5	30.762971709863	30.762972278877	5.70×10^{-07}	6.79×10^{-10}
1/3	30.762972031150	30.762972279287	2.48×10^{-07}	1.31×10^{-10}
1/4	30.762972141127	30.762972279515	1.38×10^{-07}	4.08×10^{-11}
1/5	30.762972191451	30.762972279540	8.81×10^{-08}	1.66×10^{-11}
1/6	30.762972218596	30.762972279540	6.10×10^{-08}	7.94×10^{-12}
1/7	30.762972234886	30.762972279544	4.47×10^{-08}	4.27×10^{-12}
1/8	30.762972245423	30.762972279554	3.41×10^{-08}	2.49×10^{-12}
1/9	30.762972252627	30.762972279555	2.69×10^{-08}	1.55×10^{-12}
1/10	30.762972257770	30.762972279555	2.18×10^{-08}	1.02×10^{-12}

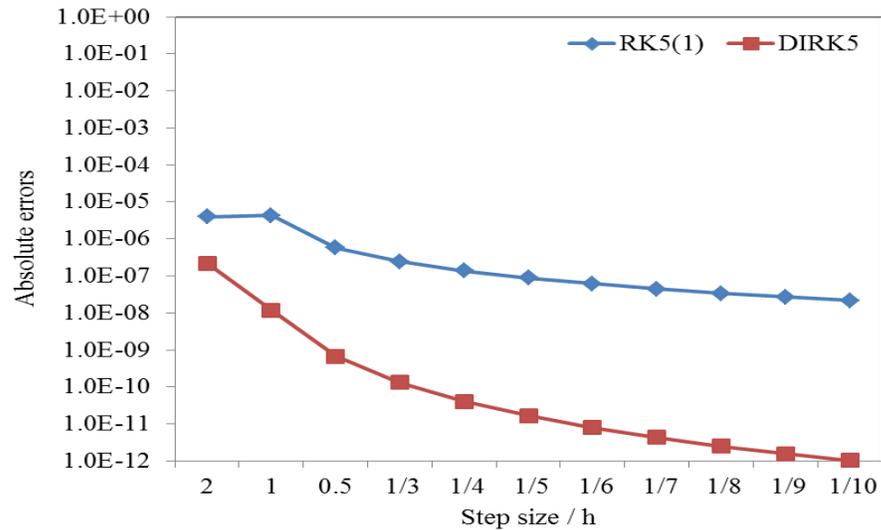


Figure 3. Absolute errors for the explicit vs implicit Runge-Kutta methods

CONCLUSION

Two methods, namely the modified explicit fifth-order Runge-Kutta method (RK5(1)) and the modified diagonally implicit Runge-Kutta method (DIRK5) were used to solve a biomechanical model of a cyclist coasting downhill. The numerical results obtained showed that DIRK5 presented a better result and accuracy when compared with RK5(1). We concluded that both methods can be utilised to obtain numerical solutions for the biomechanical model of a cyclist coasting downhill. The results showed excellent agreement between both methods and the exact solutions.

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