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Computing the First Order Polarization Tensor : Welcome BEM++!

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ABSTRACT

This paper introduce a new technique to determine the first order polarization tensor by using arecent developed software called as the BEM++. We provide a quick guideline to prepare a simple script in Python with BEM++ library to fastly compute the tensors. Two examples to highlight the convergence of the method are provided and also discussed.

Keywords: boundary integral equation, conductivity, matrix.

INTRODUCTION

The terminolgy polarization tensor (PT) arises from advance study in electromagnetics and acoustics by many authors such as Polya and Szegı (1951), Raab and de Lange (2005) and Ammari and Kang (2007). In Electrical Impedance Tomography (EIT) system, the PT is extensively studied to improve the reconstruction algorithm of conducting materials in certain spaces. Meanwhile, it is sometimes referred as the dipole in electromagnetic fields and shows charge distribution of the object. However, for some applications such as electro-sensing fish (Taufiq and Lionheart, 2012) and security screening (Marsh et al., 2013), fitting the object to its first order PT could also be very useful to understand and describe the object as it offers lower computational than reconstructing full image of the object. Furthermore, reconstructing back the image can be very expernsive due to high cost in tecnhology and expertise.

Besides using analytical definitions, several numerical methods has been developed to easily compute the related PT and can be found for examples in Capdeboscq et al. (2012) and Taufiq and Lionheart (2013). The method developed by Capdeboscq et al. (2012) so far is applicable for two-dimensional cases only while the PT obtained by following Taufiq and Lionheart (2013) is restriced to the first order PT for any three-dimensional domains. Both techniques suggest that the convergence of computation decreases when less number of meshes is used to approximate the object and also when the object has high value of conductivity.

Since objects characterized by the PT can be made from high conducting material, the need for improvement in computing the PT are then becoming very crucial especially for the case of high level of conductivity. Therefore, the main purpose of this report is to give a short introduction to the software BEM++ in computing the first order PT of three dimensional domain from its definition as boundary integral operators specifically. The software as introduced by Śmigaj et al. (2012) is choosen mainly because of its specialization in solving partial differential equations in the form of boundary integral equations. Before proceeding further, the next section will firstly review about mathematical background of the related PT. The procedure to compute the first order PT by using BEM++ as well as two examples will then follow after that.

GENERALIZED POLARIZATION TENSOR

We consider the polarization tensor (PT) originating from a transmission problem which appears in many literatures such as in Holder (2005), Ammari and Kang (2007) and Adler et al. (2011). Consider an object of Lipschitz bounded domain B in ℝ3 such that the origin *O* is in *B* and let the conductivity of *B* be equal to *k* where $0 < k \neq 1 < +\infty$. Suppose that *H* is a harmonic function in $\mathbb{R}^3$ and *u* be the solution to the

$$\begin{cases} \text{div}\big(1 + (k-1)\chi(B)\text{grad(u)}\big) = 0 \text{ in } \mathbb{R}^3 \\ u(x) - H(x) = O(|x|^2)\text{as|x|} \longrightarrow \infty \end{cases} \tag{1}$$

where χ denotes the characteristic function of *B*. The formulation (1) is actually used in many industrial applications such as medical imaging, landmine detector and material sciences (Holder, 2005; Adler et al., 2011). The PT is then defined by Ammari and Kang (2007) through the following far-field expansion of *u* as

$$(u - H)(x) = \sum_{|i|,|j|=1}^{+\infty} \frac{(-1)^{|i|}}{i!j!} \partial_x^i \Gamma(x) M_{ij}(k, B) \partial^j H(0) \text{ as } |\text{x}| \longrightarrow +\infty \tag{2}$$

for *i*,*j* multi-indices, Γ is a fundamental solution of the Laplacian and $M_{ij}(k, B)$ is the generalized polarization tensor (GPT).

Furthermore, the definition of GPT in (2) is extended again by Ammari and Kang (2007) through an integral equation over the boundary of *B* by

$$M_{ij}(k, B) = \int_{\partial B} y^j \phi_i(y) d\sigma(y) \tag{3}$$

where $\phi_i(y)$ is given by

$$\phi_i(y) = (\lambda I - K_B^*)^{-1}(\nu_x \cdot \nabla x^i)(y) \tag{4}$$

for *x*, $y \in \partial B$, *I* is the identity, ν_x is the outer unit normal vector to the boundary ∂B at *x* and $\lambda = (k+1) / 2(k-1)$. K_B^* is singular integral operator defined with the Cauchy principal value P.V by

$$K_B^* \phi(x) = \frac{1}{4\pi} \text{p. v.} \int_{\partial B} \frac{(x-y).\nu_x}{|x-y|^3} \phi(y) d\sigma(y) \tag{5}$$

for some density $\phi(y)$ and $|x - y|$ is the distance between *x* and *y*. Formulas (3), (4) and (5) isthe main interest in this study to compute the PT because of their advantage that depend only on the shape of *B* and the conductivity instead of any other parameters of the problem. Besides, if *B* is sphere centered at the origin in the Cartesian coordinate system, the first order PT of *B* denoted by *M*(*k*,*B*) at conductivity *k* is derived by Ammari and Kang (2007) as

$$M(k,B) = (k-1)|B| \begin{bmatrix} 3/(2+k) & 0 & 0 \\ 0 & 3/(2+k) & 0 \\ 0 & 0 & 3/(2+k) \end{bmatrix}. \quad (6)$$

$|B|$ here is the volume of the sphere.

FORMULATING THE FIRST ORDER POLARIZATION TENSOR IN BEM++

Recently, BEM++ is developed to devote the boundary element method (BEM) in solving partial differential equations numerically by formulating them as boundary integral equations. It is basically an open-source library which consists solutions for Laplace and Helmholtz problems in three dimensional spaces. However, it can be possibly integrated into other projects either in C++ or Python script. Any interested party may freely access it online at *www.bempp.org*. For the purpose of this study, the extensive documentation about BEM++ by Śmigaj et al. (2012) will also be referred.

The procedure to compute the first order PT for an object at specified conductivity through BEM++ here is almost similar to Taufiq and Lionheart (2013). However, several formulations in there can be neglected as they are automatically done by BEM++ later on. At this stage, the boundary of the object is firstly triangularized by the software Gmsh (see Geuzaine and Remacle (2009) and Figure 1 as examples). The nodes and elements produced by Gmsh which construct the object are then imported by BEM++ as its input. This information will be used to compute the first order PT according to (3), (4) and (5).

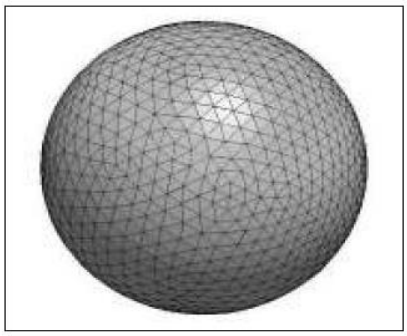

Figure 1: Triangularization of a Sphere

We will now briefly discuss how to build a Python script to compute the first order PT through BEM++. After setting all necessary modules for the script by following Śmigaj et al. (2012), the boundary integral operatorK_B^* in (5) is firstly defined as negative of Adjoint Double Layer Potential for Laplace equation. Both I and K_B^* are then set to be Boundary Operator objects in BEM++ as well as instructed to be automatically approximated and discretized by space of piecewise constant scalar functions. After that, the boundary integral operator $(\lambda I - K_B^*)$ is created at specificied conductivity k. These steps will produce $(\lambda I - K_B^*)$as $N \times N$ matrix where N is the total elements approximating the object.

Now, notice that for the first order PT, $\left(v_x \cdot \nabla x^i\right)(y)$ are the *i*-th, *j*-th and *k*-th component of the outer unit normal vector to the boundary of the object at arbitrary point x namely v_x. This means every $\phi_i(y)$ for the first order PT can be obtained by multiplying every component of the normal vector by $(\lambda I - K_B^*)^{-1}$ as given in (4). In order to achieve this, the normal vector is firstly defined as a Grid Function in BEM++ and approximated at each element of the boundary by a piecewise constant scalar functions. Using these approximations, instead of original, the system

$$(\lambda I - K_B^*)\phi_i(y) = v_x \tag{7}$$

is then solved for $\phi_i(y)$ by using the function `solver' in BEM++. This produces three $\phi_i(y)$ where each $\phi_i(y)$ is approximated and evaluated at every element of the boundary.

Finally, to determine the first order PT through (3), y^j is firstly restricted to be 1-st, 2-nd and 3-rd coordinate of *y*. The first order PT then can be obtained by integrating the product between every coordinate of y and every $\phi_i(y)$ over the boundary of the object. This integral is approximated by the means of quadrature rule through BEM++ as in Taufiq and Lionheart (2013) and results the first order PT of an object at specified conductivity to be a 3×3 matrix.

EXAMPLES AND DISCUSSION

The first order PT for sphere of radius 1 and centered at origin at conductivity 1.5 and 10000 are computed according to analytical formula (6) as

$$\begin{bmatrix} 1.7952 & 0 & 0 \\ 0 & 1.7952 & 0 \\ 0 & 0 & 1.7952 \end{bmatrix} \text{ and } \begin{bmatrix} 12.5626 & 0 & 0 \\ 0 & 12.5626 & 0 \\ 0 & 0 & 12.5626 \end{bmatrix}$$

Our previous computation for both cases based on Taufiq and Lionheart (2013) through MATLAB with 9920 elements triangularizing the sphere gives

$$\begin{bmatrix} 1.7890 & 0 & 0.0001 \\ 0 & 1.7890 & 0 \\ 0.0001 & 0 & 1.7890 \end{bmatrix} \text{ and } \begin{bmatrix} 12.3486 & 0.0011 & 0.0023 \\ 0.0011 & 12.3486 & 0.0009 \\ 0.0023 & 0.0009 & 12.3486 \end{bmatrix}$$

However, these results are then improved by using BEM++ with 46184 elements and are given by

$$\begin{bmatrix} 1.7947 & 0 & 0 \\ 0 & 1.7947 & 0 \\ 0 & 0 & 1.7947 \end{bmatrix} \text{ and } \begin{bmatrix} 12.5579 & 0 & 0 \\ 0 & 12.5579 & 0 \\ 0 & 0 & 12.5579 \end{bmatrix}$$

As we can see from the above matrices, the results from BEM++ improve the computation to three decimal places for conductivity 1.5 and to two decimal places for conductivity 10000. This is probably due to larger total of elements used to approximate the sphere in BEM++. Our past experience suggest that BEM++ could be efficiently built by its

developer to allocate the memory to the processor during computation since our past program in MATLAB stop to run for more than 15000 elements.

Next, the first order PT for cube of size 1×1×1 where two of its vertices are at origin and (1,1,1) is computed also at conductivity 1.5 and 10000 with BEM++ after the cube is approximated by 19406 elements. There is no analytical solution for the cube so far but because cube is best fitted to a sphere, it is expected the first order PT for cube to has only one eigenvalue as suggested by Taufiq and Lionheart (2012). This means three eigenvalues of the first order PT for cube must be equal. The results obtained show that three eigenvalues of the first order PT for the cube at conductivity 1.5 are all equal to 0.4305 while the eigenvalues at conductivity 10000 are equal to 3.6307, 3.6304 and 3.4771 respectively. This means the computation is still far from converge for the case of high value of conductivity.

SUMMARY

This paper has briefly summarized on how BEM++ is used to improve numerically the computation of the first order polarization tensor where two examples are included as well. Our commitment with the software at this stage is still preliminary and we are still exploring the software to look for other possible ways to improve the computation. Once better results are obtained, we are hoping to improve our implementation of the PT to the related applications in the future.

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