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Expansion of $\cos^n\theta$ and $\sin^n\theta$ in terms of cosines and sines of multiple of θ by using Noor's triangle

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ABSTRACT

In the present study we introduced a triangle, like Pascal's triangle, to obtain directly the coefficients in the expansion of $\cos^n\theta$ and $\sin^n\theta$ in terms of cosines and sines of multiple of θ, where $\boldsymbol{n}$ is any positive integer. This type of expansion is given by Datta et al. in which they have used the binomial theorem in order to obtain the coefficients therein. We name this triangle as "Noor's triangle".

Keywords: Pascal's triangle, Binomial coefficients, De Moivre's theorem.

INTRODUCTION

In this work we constructed a triangle, like the Pascal's triangle. The i^{th} entry of the n^{th} row gives the i^{th} coefficient in the expansion of $\cos^n\theta$ or $\sin^n\theta$ in terms of cosines and sines of multiples of θ.

Browein and Bailey (2003) given that if

$$x = \cos\theta + i\sin\theta \tag{1}$$

then

$$\frac{1}{x} = \cos\theta - i\sin\theta \tag{2}$$

De Moivre's theorem gives

$$x^n = \cos n\theta + i\sin n\theta \tag{3}$$

and

$$\frac{1}{x^n} = \cos n\theta - i\sin n\theta \tag{4}$$

Adding Equation (1) and (2) and taking the n^{th} power, one can obtain

$$
\begin{aligned}
(2\cos\theta)^n &= (x+\frac{1}{x})^n \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}\left(\frac{1}{x}\right)^k \\
&= x^n + \binom{n}{1}x^{n-1}\left(\frac{1}{x}\right)+\binom{n}{2}x^{n-2}\left(\frac{1}{x}\right)^2+\ldots+\binom{n}{n}x^{n-n}\left(\frac{1}{x}\right)^n \\
&= x^n + \binom{n}{1}x^{n-2}+\binom{n}{2}x^{n-2}\left(\frac{1}{x}\right)^2+\ldots+\binom{n}{n-1}x^{n-(n-1)}\left(\frac{1}{x}\right)^{n-1}+\left(\frac{1}{x}\right)^n \\
&= \binom{n}{0}(x^n+\frac{1}{x^n})+\binom{n}{1}(x^{n-2}+\frac{1}{x^{n-2}})+\ldots+\frac{1}{2}\binom{n+1}{\frac{n+1}{2}}\left(x+\frac{1}{x}\right). \qquad (5a) \\
&\quad (\text{when } n \text{ is odd}) \\
&= \binom{n}{0}(x^n+\frac{1}{x^n})+\binom{n}{1}(x^{n-2}+\frac{1}{x^{n-2}})+\ldots+\frac{1}{2}\binom{n+1}{\frac{n+1}{2}}. \qquad (5b) \\
&\quad (\text{when } n \text{ is even})
\end{aligned}
$$

Adding Equation (3) and (4) we get

$$(x^n+\frac{1}{x^n}) = 2\cos n\theta.$$

and hence $(x+\frac{1}{x}) = 2\cos\theta$, $(x^2+\frac{1}{x^2}) = 2\cos 2\theta$, ..., $(x^n+\frac{1}{x^n}) = 2\cos n\theta$. Using these values in Equation $(5a)$ and $(5b)$, we obtain respectively

$$2^{n-1}\cos^n\theta = \cos n\theta + \binom{n}{1}\cos((n-2)\theta)+\binom{n}{2}\cos((n-4)\theta)+\ldots+\frac{1}{2}\binom{n+1}{\frac{n+1}{2}}\left(x+\frac{1}{x}\right)$$

and

$$2^{n-1}\cos^n\theta = \cos n\theta + \binom{n}{1}\cos((n-2)\theta)+\binom{n}{2}\cos((n-4)\theta)+\ldots+\frac{1}{2}\binom{n+1}{\frac{n+1}{2}}.$$

When n is odd then by similar method, we obtain

$$(2i)^{n-1}\sin^n\theta = \sin n\theta - \binom{n}{1}\sin((n-2)\theta)+\binom{n}{2}\sin((n-4)\theta)+\ldots\text{until the coefficient of } \theta \text{ is } 1,$$

and when n is an even integer, then

$$(2i)^{n-1}\sin^n\theta = \cos n\theta - \binom{n}{1}\cos((n-2)\theta) + \binom{n}{2}\cos((n-4)\theta) + ... - \frac{1}{2}\binom{n+1}{\frac{n+1}{2}}.$$

In Section 2, *Noor's triangle* is constructed along with its complete justification. The main results are stated and the algorithm developed in C++ , are given to generate the said triangle. At the end some special properties of this triangle are discussed.

CONSTRUCTION OF THE NOOR'S TRINGLE

The methodology is as follows:

$$\begin{array}{lllll}
1 \\
1 & 3 \\
1 & a_{32} \\
1 & a_{42} & a_{43} \\
1 & a_{52} & a_{53} \\
1 & a_{62} & a_{63} & a_{64} \\
1 & a_{72} & a_{73} & a_{74} \\
1 & a_{82} & a_{83} & a_{84} & a_{85} \\
1 & a_{92} & a_{93} & a_{94} & a_{95}
\end{array}$$

Here

$$a_{ij} = \begin{cases} 2a_{(i-2)(j-1)} + a_{(i-1)(j-1)} & \text{if } a_{ij} \text{ is the first entry of the } j^{th} \text{ column} \\ a_{(i-1)(j-1)} + a_{(i-1)j} & \text{if } a_{ij} \text{ is not the first entry of the } j^{th} \text{ column} \end{cases}.$$

In the following we give some useful results which will be very helpful for the expansion of $\cos^n\theta$ and $\sin^n\theta$ in terms of cosines and sines of multiple of θ.

Theorem 1: If n is any even integer, then
$2^n\cos^{n+1}\theta = a_{n1}\cos((n+1)\theta) + a_{n2}\cos(((n+1)-2)\theta) + a_{n3}\cos(((n+1)-4)\theta) + ... + a_{n(n-2)}\cos\theta$.

Proof:

If $n = 2$, then

$$2^2\cos^3\theta = a_{21}\cos 3\theta + a_{22}\cos\theta$$

so it is true for $n = 2$.

Next we suppose it is true for $n = k$, then

$$2^k \cos^{k+1}\theta = a_{k1}\cos(k+1)\theta + a_{n2}\cos\big((k+1)-2\big)\theta + a_{n3}\cos\big((k+1)-4\big)\theta + \ldots + a_{k(k-2)}\cos\theta.$$

Now if we check it for $n = k+1$. By putting $n = k+1$, we get

$$2^{k+1}\cos^{(k+1)+1}\theta = a_{(k+1)1}\cos\big((k+1)+1\big)\theta + a_{(k+1)2}\cos\big(((k+1)+1)-2\big)\theta + \ldots + a_{(k+1)((k+1)-2)}\cos\theta.$$

that is

$$2^{k+1}\cos^{k+2}\theta = a_{(k+1)1}\cos\big((k+2)\theta + a_{(k+1)2}\cos\big((k+2)-2\big)\theta + \ldots + a_{(k+1)(k-1)}\cos\theta.$$

This completes the proof

Similarly one can prove the following results.

Theorem 2: If n is any odd integer, then

$$2^n \cos^{n+1}\theta = a_{n1}\cos\big((n+1)\theta\big) + a_{n2}\cos\big(((n+1)-2)\theta\big) + a_{n3}\cos\big(((n+1)-4)\theta\big) + \ldots + \tfrac{1}{2}\binom{n+1}{\frac{n+1}{2}} .$$

Theorem 3: If n is any even integer then,

$$(2i)^n \sin^{n+1}\theta = a_{n1}\sin\big((n+1)\theta\big) - a_{n2}\sin\big(((n+1)-2)\theta\big) + a_{n3}\sin\big(((n+1)-4)\theta\big) - \ldots - a_{n(n-3)}\sin\theta$$

.

Theorem 4: If n is any odd integer then,

$$(2)^n (i\sin\theta)^{n+1} = a_{n1}\cos\big((n+1)\theta\big) - a_{n2}\cos\big(((n+1)-2)\theta\big) + a_{n3}\cos\big(((n+1)-4)\theta\big) - \ldots - \tfrac{1}{2}\binom{n+1}{\frac{n+1}{2}} ,$$

where $a_{n1} = 1$ *for all* n .

JUSTIFICATION

In this section we give some examples to expand $\cos^n\theta$ and $\sin^n\theta$ in terms of cosines and sines of multiple of θ by using Noor's triangle. If we take $n = 6$ then we have our triangle as

1
1 3
1 4
1 5 10
1 6 15
1 7 21 35

From Theorems (1) and (3) we obtain

$$2^6\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$$

and

$$-2^6\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta.$$

If we take $n=5$ then we have our triangle as

1		
1	3	
1	4	
1	5	10
1	6	15

From Theorems (2) and (4) we obtain

$$2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

and

$$-2^5\sin^6\theta = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10.$$

ALGORITHM

The researchers have designed the following algorithm in standard C++. By this algorithm we can construct our triangle any value of $n = 1,2,3,...$.

```
#include<conio.h>
#include<iostream>
#include<iomanip>
using namespace std;
const int space = 10;
void main()
{
int lines;
long arrayeven[40];
long arrayodd[40];
arrayeven[0]= 1;
arrayodd[0]=1;
arrayodd[1]=3;
int evencount=1;
int oddcount=1;
```

```
cout<<"Enter numbers of lines to print:";
cin>>lines;
if(lines==1)
cout<<endl<<" 1";
if(lines>=2)
cout<<endl<<" 1"<<endl
  <<" 1"<<setw(space)<<"3";
for(int i=2;i<lines;i++)
{
if(i%2!=0)
{
int lastoddvalue = arrayodd[oddcount];
cout<<endl<<" 1";
for(int k=0;k<evencount;k++)
{
arrayodd[k+1]=arrayeven[k]+arrayeven[k+1];
cout<<setw(space)<<arrayodd[k+1];
}
oddcount++;
arrayodd[oddcount]=(lastoddvalue*2)+arrayeven[evencount];
cout<<setw(space)<<arrayodd[oddcount];
evencount++;
}
else
{
cout<<endl<<" 1";
for(int j=0;j<oddcount;j++)
{
arrayeven[j+1]=arrayodd[j]+arrayodd[j+1];
cout<<setw(space)<<arrayeven[j+1];
}
}
}
getche();
cout<<endl;
}
```

PROPERTIES OF NOOR'S TRINGLE

i. If the second number of each row are looked at and going from last row in the upward direction to the first row, it can be seen that these numbers are the coefficients of θ.

ii. Another interesting property of this triangle is that, in rows where the second number is prime, all the numbers in that row are multiples of that prime number except the number 1.

iii. For any entry a_{ij} of the Noor's triangle $i \geq j$.

CONCLUSION

Expansion of $\cos^n\theta$ and $\sin^n\theta$ in terms of cosines and sines of multiple of θ plays an important role in the study of trignometric functions. The method used by Datta et al for this type of expansion contains very long calculations for example if $n = 30$, then it will take a long time to expand $\left(x + \frac{1}{x}\right)^{30}$. The method we introduced in this paper is very quick and easy no matter how large is n.

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