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Assorted Iterative Methods as Solution of Fuzzy Linear Systems

Nurhakimah Ab Rahman and Lazim Abdullah

Department of Mathematics, Faculty of Sciences and Technology, University Malaysia Terengganu, 21030 Kuala Terengganu, Terengganu, Malaysia

ABSTRACT

Solutions of fuzzy linear systems can be obtained by some of iterative methods. However, these methods have distinct algorithms and come with different number of iterations and parameters. This paper presents assorted iterative methods as possible solutions for a general $n \times n$ fuzzy system of linear equations in the form $Ax = b$. Six different iterative methods were discussed. Numerical example for every iterative method was given to illustrate the method. It is hoped that this review provides an input in the study of solution of fuzzy linear systems.

Keywords: linear sysetem linear systems, fuzzy numbers, iterative methods, extrapolation parameters

INTRODUCTION

One of the most important discussions in applied mathematics is system of linear equations. The systems are germane for modeling and solving in many branches of knowledge areas such as physics, statistics, engineering, economics, finance and even social sciences. In most of the applications, parameters of the system are represented by crisp. However, there are cases where parameters represented by fuzzy numbers. This situation is true especially when estimation of the system coefficients is imprecise and only some vague knowledge about the actual value of the parameters is available. It has been suggested that the system may be convenient to represent some or all of them with fuzzy numbers. The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh (1975) and Chang and Zadeh(1972). It was then further developed in 1990s when Buckley and Qu (1999) investigated the system and subsequently Friedman e al. (1998) proposed a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system.

As fuzzy numbers steadily matured in linear systems, there were many researchers proposed various methods to solve fuzzy linear system of equations. For example, Asadi et al. (2005) developed a method for solving a $m \times n$ fuzzy linear system for $m \leq n$. They solved $m \times n$ fuzzy linear system by a $2m \times 2n$ crisp function linear system. Matinfar et al. (2008) solved rectangular fuzzy linear system of equations based on matrix decomposition method. They dealt with Householder process to obtain a QR-decomposition for the coefficient matrix of the extended linear system of the fuzzy linear system equation. Another decomposition method that is Adomian decomposition method was used by Allahviranloo (2004) to solve fuzzy linear equation. Hence it is immensely important to further examine numerical procedures that would appropriately treat general fuzzy linear systems and solve them.

Diverse numerical solutions for fuzzy linear systems have been explored by many scholars Friedman e al. (1998), Matinfar et al. (2008), Allahviranloo and Kermani (2006), Dehgan and Hashemi (2006), Kandel et al. (1996), Ma et al. (2000) and Young (1971). Generally, there are two classes of methods for solving linear equations. It is direct method and iterative method. A direct method is a method that in exact arithmetic will yield the exact solution using a finite number of steps. Iterative method, on the other hand, produces a sequence of approximate solutions that converges to the exact solution in the limit. Based on some previous studies, fuzzy linear systems are mostly solved by iterative methods. Iterative method for solving fuzzy linear systems $Ax = b$ begins with initial guess for solution and successively improves it until solution is as accurate as desired. In theory, infinite number of iterations might be required to converge to exact solution. In practice, iteration terminates when residual $\|b - Ax\|$, or some other measure of error is as small as desired. Iterative methods are especially useful when matrix A sparse because no fill is incurred. Therefore it is important to unveil various iterative numerical procedures or algorithms that would solve fuzzy linear systems. Therefore the purpose this paper is to review assorted iterative methods in solving fuzzy linear systems.

This paper is organized in the following manner. Basic definitions on fuzzy numbers and fuzzy linear system are given in Section 2. In Section 3, iterative methods to solve problem of fuzzy linear systems are presented. Numerical examples to illustrate the iterative methods are given Section 4. Section 5 ends this paper with conclusion.

PRELINIMARIES

In this section, fuzzy linear systems, and fuzzy numbers are defined.

Definition 2.1 Fuzzy linear systems

Definitions of fuzzy linear systems have been provided by Friedman e al. (1998), Asadi et al. (2005), Allahviranloo (2004), Allahviranloo and Kermani (2006) and Ma et al. (2000) as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= y_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= y_n \end{aligned} \tag{2.1}$$

The $n \times n$ matrix form of the above equations is $AX = Y$, where the coefficient matrix is a crisp $n \times n$ matrix and $y_i \in E^1, 1 \le i \le n$. This system is called a fuzzy linear system.

Definition 2.2: Fuzzy numbers Asadi et al. (2005), Allahviranloo (2005), Kandel et al. (1996) and Ma et al. (2000) .
A fuzzy number is a fuzzy set $u : R^1 \to I = [0,1]$ which satisfies

i. u is upper semicontinuous.

ii. $u(x) = 0$ outside some interval $[c, d]$.

iii. There are real numbers $a, b : c \le a \le b \le d$ for which

a) $u(x)$ is monotonic increasing on $[c, a]$.

b) $u(x)$ is monotonic decreasing on $[b, d]$.

c) $u(x) = 1, a \le x \le b$.

The set of all the fuzzy numbers is denoted by E^1.

The two definitions become profound concepts in exploring iterative methods in solving fuzzy linear systems.

ITRATIVE METHODS TO SOLVE FUZZY LINEAR SYSTEMS

In this section, some iterative methods were discussed to perform several techniques in solving fuzzy linear systems.

Jacobi method

Allahviranloo (2004) introduced the Jacobi method to solve fuzzy linear system for the first time. Jacobi iterative technique can be formulated as follows:

$$\begin{aligned} \underline{X}^{(k+1)} &= D_1^{-1}\underline{Y} - D_1^{-1}(L_1 + U_1)\underline{X}^k - D_1^{-1}S_2\overline{X}^k \\ \overline{X}^{(k+1)} &= D_1^{-1}\overline{Y} - D_1^{-1}(L_1 + U_1)\overline{X}^k - D_1^{-1}S_2\underline{X}^k \end{aligned} \qquad (3.1)$$

For the Jacobi method, the sequence $\{X^{(k)}\}$ is easily computed and the rate of convergence is better than the Richardson's method [10]. Notes that, Jacobi iterates are converge to the unique solution $X = A^{-1}Y$ for any X^0, where $X \in R^{2n}$ and $(\underline{X}, \overline{X}) \in E^n$ Allahviranloo and Kermani (2006).

Gauss Seidel method

Allahviranloo (2004) also proposed Gauss Seidel method as one of the iterative methods to solve fuzzy linear system. There are forward and backward Gauss Seidel. Hence, Gauss Seidel iterative technique was formulated as follows:

$$\begin{aligned}\underline{X}^{(k+1)} &= (D_1+L_1)^{-1}\underline{Y}-(D_1+L_1)^{-1}U_1\underline{X}^{k}-(D_1+L_1)^{-1}S_2\overline{X}^{k}\\ \overline{X}^{(k+1)} &= (D_1+L_1)^{-1}\overline{Y}-(D_1+L_1)^{-1}U_1\overline{X}^{k}-(D_1+L_1)^{-1}S_2\underline{X}^{k}\end{aligned} \tag{3.2}$$

Successive over relaxation (SOR) method

Allahviranloo (2005) has recently developed SOR method as a new iterative method in solving fuzzy linear system. SOR method is a modification of the Gauss Seidel iteration.
Hence, the SOR iterative method is written as follows:

$$\begin{aligned}\underline{X}^{(k+1)} &= (I+\omega L_1)^{-1}\omega D^{-1}\underline{Y}-(I+\omega L_1)^{-1}[(1-\omega)I+\omega U_1]\underline{X}^{(k)}-(I+\omega L_1)^{-1}\omega S_2\overline{X}^{(k)}\\ \overline{X}^{(k+1)} &= (I+\omega L_1)^{-1}\omega D^{-1}\overline{Y}-(I+\omega L_1)^{-1}[(1-\omega)I+\omega U_1]\overline{X}^{(k)}-(I+\omega L_1)^{-1}\omega S_2\underline{X}^{(k)}\end{aligned} \tag{3.3}$$

However, SOR method needs extrapolation parameter in its formula where if the SOR method be converge, then $0<\omega<2$. For choice of ω with $0<\omega<1$, this method can be used to obtain convergence of some systems that are not convergent by the Gauss Seidel method. For choice of ω with $\omega>1$, this method is used to accelerate the convergence for systems that are convergent by the Gauss Seidel technique Allahviranloo and Kermani (2006). Evidently, SOR method is reduced to the Gauss Seidel method for $\omega=1$.

Richardson method

Dehgan et al.(2006) proposed Richardson method as an iterative method to solve fuzzy linear system. Hence, the Richardson method will be in the following form:

$$\begin{aligned}\underline{X}^{(k+1)} &= \underline{Y}+(I_n-B)\underline{X}^{(k)}+C\overline{X}^{(k)}\\ \overline{X}^{(k+1)} &= \overline{Y}+(I_n-B)\overline{X}^{(k)}+C\underline{X}^{(k)}\end{aligned} \tag{3.4}$$

In this approach, the sequence $\{X^m\}$ is easily computed, but the rate of convergence of the sequence $\{X^m\}$ is very slow. By this method, $\rho(Q_{RICH})=\max\{|1-m(S)|,|1-M(S)|\}$. So when S is a symmetric positive definite matrix, then the Richardson method converges if and only if $M(S)<2$. However, fuzzy linear systems are divergent to the solution Dehgan et al (2005).

Jacobi over Relaxation (JOR) method

Based on Dehgan et al (2006). Jacobi over Relaxation method is an extrapolated of Jacobi method which firstly performed by Young [11]. Hence, the JOR iterative method will be

$$\begin{cases} \underline{X}^{(m+1)} = \omega D_1^{-1}\underline{Y} - \omega D_1^{-1}\left[\left(1-\dfrac{1}{\omega}\right)D_1 + L_1 + U_1\right]\underline{X}^{(m)} + \omega D_1^{-1} C\,\overline{X}^{(m)} \\ \overline{X}^{(m+1)} = \omega D_1^{-1}\overline{Y} - \omega D_1^{-1}\left[\left(1-\dfrac{1}{\omega}\right)D_1 + L_1 + U_1\right]\overline{X}^{(m)} + \omega D_1^{-1} C\,\underline{X}^{(m)} \end{cases} \tag{3.5}$$

Evidently the JOR method is reduced to Jacobi method for $\omega = 1$ [6].

The accelerated over relaxation (AOR) method

The AOR method is first introduced by Hadjidimos (1978). Then, Dehgan et al. (2006) used this method for solving a fuzzy linear system by using two parameters. There are relaxation parameter (α) and extrapolation parameter (ω).

Hence, the AOR method is

$$\begin{aligned} \underline{X}^{(m+1)} &= \omega(D_1+\alpha L_1)^{-1}\underline{Y} + (D_1+\alpha L_1)^{-1}\left[(1-\omega)D_1 + (\alpha-\omega)L_1 - \omega U_1\right]\underline{X}^{(m)} \\ &\quad - \omega(D_1+\alpha L_1)^{-1} C\,\overline{X}^{(m)} \\ \overline{X}^{(m+1)} &= -\alpha\omega(D_1+\alpha L_1)^{-1}\underline{Y} + \omega(D_1+\alpha L_1)^{-1}\overline{X}^{(m)} + (D_1+\alpha L_1)^{-1} \\ &\quad \left[(1-\omega)D_1 + (\alpha-\omega)L_1 - \omega U_1 + \alpha\omega C(D_1+\alpha L_1)^{-1}C\right]\overline{X}^{(m)} \\ &\quad - \omega(D_1+\alpha L_1)^{-1}C(D_1+\alpha L_1)^{-1}\left[(1-\alpha)D_1 - \alpha U_1\right]\underline{X}^{(m)} \end{aligned} \tag{3.6}$$

Evidently, the forward AOR method is reduced to

i. Jacobi method for $\alpha = 0$ and $\omega = 1$.
ii. JOR method for $\alpha = 0$.
iii. Forward Gauss Seidel method for $\alpha = \omega = 1$.
iv. EGS method for $\alpha = 1$ and $\omega = \alpha$.
v. Forward SOR method for $\alpha = \omega$.

NUMERICAL EXAMPLES

As to illustrate the methods in Section 3, the following examples are presented. However details of the solutions are not discussed in this section.

Jacobi method

Allahviranloo (2004) has applied Jacobi method in the following example.

Consider the 2×2 fuzzy system

$x_1 - x_2 = (r, 2-r)$

$x_1 + 3x_2 = (4+r, 7-2r)$

By using formula $SX = Y$ which be first proposed by Friedman [7],

$$X = \begin{bmatrix} \underline{x}_1(r) \\ \underline{x}_2(r) \\ -\overline{x}_1(r) \\ -\overline{x}_2(r) \end{bmatrix} \text{ and } S^{-1}Y = \begin{bmatrix} 1.125 & -0.125 & 0.375 & -0.375 \\ -0.375 & 0.375 & -0.125 & 0.125 \\ 0.375 & -0.375 & 1.125 & -0.125 \\ -0.125 & 0.125 & -0.375 & 0.375 \end{bmatrix} \begin{bmatrix} r \\ 4+r \\ r-2 \\ 2r-7 \end{bmatrix}.$$

Hence, the solution obtained is as follows:

$x_1 = (1.375 + 0.625r,\ 2.875 - 0.875r)$

$x_2 = (0.875 + 0.125r,\ 1.375 - 0.375r)$

Gauss Seidel method

Allahviranloo (2004) also applied Gauss Seidel method to the example below. Consider the 3×3 fuzzy system

$x_1 + x_2 - x_3 = (r, 2-r),$

$x_1 - 2x_2 + x_3 = (2+r, 3),$

$2x_1 + x_2 + 3x_3 = (-2, -1-r)$

Formula (3.2) is used to solve this systems and the solution is as follows:

$x_1 = (0.1399r - 0.4125,\ -0.3217r + 0.0351)$

$x_2 = (0.2894r + 0.9125,\ 0.0970r + 1.1076)$

$x_3 = (-0.1513r - 0.7353,\ -0.1897r - 0.6969)$

Successive over relaxation (SOR) method

Allahviranloo and Kermani (2006) also used the same numerical example as in Jacobi method to solve fuzzy linear systems by SOR method. The solution is as follows:

$x_1 = (1.375 + 0.625r,\ 2.875 - 0.875r)$

$x_2 = (0.875 + 0.125r,\ 1.375 - 0.375r)$

By this numerical method, when extrapolation parameter $\omega = 1.0$, SOR method reduced to the Gauss Seidel method. Besides, the system converges with different number of iterations when different extrapolation parameter is used.

Richardson method

Dehgan et al. (2006) applied the numerical example as below to solve systems by using Richardson method.

$$3x_1 - x_2 = (10r, 20 - 10r)$$
$$-x_1 + 3x_2 - x_3 = (6 + 2r, 10 - 2r)$$
$$-x_2 + 3x_3 = (5 + 10r, 20 - 5r)$$
$$-x_4 + 3x_5 - x_6 = (10r, 30 - 20r)$$
$$-x_5 + 3x_6 = (10 + 20r, 60 - 30r)$$

As in formula (3.4), the systems are solved by Richardson method. However, the systems not converge to the solution. So, there are no solution for this systems when be solved by Richardson method.

4.5 Jacobi over Relaxation (JOR) method

Dehgan et al. (2006) solve the below example to test the efficiency of JOR method.

$$8x_1 + 2x_2 + x_3 - 3x_5 = (r, 2 - r)$$
$$-2x_1 + 5x_2 + x_3 - x_4 + x_5 = (4 + r, 7 - 2r)$$
$$x_1 - x_2 + 5x_3 + x_4 + x_5 = (1 + 2r, 6 - 3r)$$
$$-x_3 + 4x_4 + 2x_5 = (1 + r, 3 - r)$$
$$x_1 - 2x_2 + 3x_5 = (3r, 6 - 3r)$$

By using formula (3.5), the systems are solved by JOR method. By using extrapolation parameter $\omega = 0.80$, following results is produced.

$$x_1 = (0.73086 - 0.33205r, 0.043532 + 0.35528r)$$
$$x_2 = (0.61335 + 0.16681r, 1.0774 - 0.29725r)$$
$$x_3 = (0.12562 + 0.29088r, 0.91807 - 0.50156r)$$
$$x_4 = (0.24292 - 0.33314r, -0.41827 + 0.32805r)$$
$$x_5 = (0.47448 + 0.91251r, 2.3943 - 1.0073r)$$

Accelerated over Relaxation (AOR) method

Dehgan et al. (2006) use the same numerical example above to test accelerated over relaxation method in fuzzy linear systems. Solutions using the AOR method with $\alpha = 0.60$ and $\omega = 1.333$ is obtained as follows:

$$
\begin{aligned}
x_1 &= (0.72908 - 0.33089r, 0.043622 + 0.35435r) \\
x_2 &= (0.61422 + 0.16609r, 1.0772 - 0.29676r) \\
x_3 &= (0.12628 + 0.29068r, 0.91861 - 0.50155r) \\
x_4 &= (0.24415 - 0.33344r, -0.41751 + 0.32768r) \\
x_5 &= (0.4742 + 0.91341r, 2.3957 - 1.008r)
\end{aligned}
.
$$

CONCLUSION

In this paper, some iterative methods have been reviewed to solve fuzzy linear systems. One of the more significant findings to emerge from this study is that iterative methods are properly works for fuzzy linear systems. These reviews reaffirm the feasibility of several numerical techniques in solving fuzzy linear systems. It would be interesting if further research could be undertaken to explore the possibility of other iterative methods and direct methods in solving fuzzy linear systems.

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