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A Note on Some Generalizations of Continuity in Topological Spaces

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ABSTRACT

The aim of this paper is to study some generalized continuities in topological spaces. Basic characterizations and properties concerning contra continuity, semi-continuity, precontinuity, α-continuity, *A*-continuity, *LC*-continuity, almost continuity, and weakly continuity are obtained. The relationships between generalized continuities are discussed and several counterexamples are provided to illustrate the connection of these generalized continuities.

Keywords: contra-continuous, semi-continuous, precontinuous, almost continuous.

INTRODUCTION

Continuity has been intensively studied in the field of Topology and other several branches in mathematics. Furthermore, it is an important concept that has been widely studied by many researchers because of its importance in most of applications in topological model, data modeling, engineering, sciences, economic, business, etc. The first step of generalizing continuous function was done by Levine (1961). He introduced and investigated two weakened forms of continuity in topological space and showed that the combination of weak continuity and weak* continuity characterized continuity in the most general cases, whilst semi-open sets and semi-continuity was introduced by Levine (1963).

The concept of α-continuity and α-open mappings has been introduced and studied by Mashhour et al. (1982). Mashhour et al. said that a mapping $f: X \to Y$ is α-continuous if the inverse image of each open set of Y is an α-set in X. After that, Ganster and Reilly (1989,1990) introduced locally closed sets and *LC*-continuous functions, and they gave a decomposition of continuity which provides the relationship between *A*-continuity, *LC*-continuity, α-continuity, precontinuity and semi-continuity. Another generalization of continuity called contra continuity which is a stronger form of *LC*-continuity has been found by Dontchev (1996). Moreover Kılıçman and Salleh (2006) studied and obtained some properties in general cases concerning composition of (δ-*pre*, *s*)-continuous functions under specific conditions, where the composition would yield a (δ-*pre*, *s*)-continuous function.

In this paper, we obtain the generalizations of continuity in topological spaces extended from Ganster and Reilly (1990) which involved almost continuity, weakly continuity and contra continuity. We apply all these generalized continuities to discover the relationships of the generalized continuities in topological spaces.

PRELIMIRIES

Throughout this paper, spaces (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces. Let S be a subset of a topological space (X,τ). The closure and interior of S in (X,τ) is denoted by $\mathrm{cl}(S)$ and $\mathrm{int}(S)$ respectively. Now we introduced some basic notions and results that are used in the sequel.

Definition 2.1. *A subset S of* (X,τ) *is called*

(a) *a semi-open* [10] *set if* $S \subseteq \mathrm{cl}(\mathrm{int}(S))$

(b) *a preopen* [11] *set if* $S \subseteq \mathrm{int}(\mathrm{cl}(S))$

(c) *an* α *-set* [14] *if* $S \subseteq \mathrm{int}\left(\mathrm{cl}(\mathrm{int}(S))\right)$

(d) *an A-set* [6] *if S = U ∩ F where U is open and F is regular closed*

(e) *locally closed* [5] *if* $S = U \cap F$ *where U is open and F is closed*

(f) *regular open* [3] *if* $S = \mathrm{int}(\mathrm{cl}(S))$.

The family of all *A*-sets (resp. regular closed, locally closed, semi-open and preopen) sets of (X,τ) is denoted by $A(X,\tau)$ (resp. $RC(X,\tau)$, $LC(X,\tau)$, $SO(X,\tau)$ and $PO(X,\tau)$). The complement of semi-open (resp. preopen, regular open and α -set) set is called semi-closed (resp. preclosed, regular closed and α -closed) set.

In this section, we give some definitions for the some generalized continuities in topological spaces which involved in this paper.

Definition 2.2. *A function $f: X \to Y$ is called contra-continuous* [4] (*resp. semi-continuous* [10], *precontinuous* [6], *α-continuous* [12], *A-continuous* [6], *LC-continuous* [5] *if the inverse image of each open set in Y is a closed set* (*resp. semi-open, preopen, α-set, A-set, locally closed*) *set in X.*

Definition 2.3. [16] *A function $f: X \to Y$ is said to be almost continuous* (*resp. weakly continuous*) *if for each open set V in Y containing $f(x)$, there is an open set U in X containing x such that $f(U) \subseteq \mathrm{int}(\mathrm{cl}(V))$* (*resp.* $f(U) \subseteq \mathrm{cl}(V)$).

ON GENERALIZED CONTINUTIES

As we all know, a function $f: X \to Y$ is said to be continuous if the inverse image of each open set in Y is an open in X. In this section, we gave some definitions and properties of the generalized continuities in topological spaces.

Definition 3.1. [2] *Let S be a subset of a space (X, τ). The set $\bigcup\{U \in \tau: S \subseteq U\}$ is called the kernel of S and is denoted by* $\ker(S)$.

Lemma 3.2. [7] *The following properties hold for subsets S, T of a space X;*

(a) $x \in \ker(S)$ *if and only if* $S \cap F \neq \emptyset$ *for any closed set F in X containing x.*
(b) $S \subseteq \ker(S)$ *and* $S = \ker(S)$ *if S is open in X.*
(c) *If* $S \subseteq T$, *then* $\ker(S) \subseteq \ker(T)$.

Theorem 3.3. *For the following conditions of a function $f: X \to Y$:*

(a) *f is contra-continuous*;
(b) $f^{-1}(V)$ *is open in X for every closed set V in Y*;
(c) $f(\mathrm{cl}(S)) \subseteq \ker f(S)$ *for every subset S of X*;
(d) $\mathrm{cl}\left(f^{-1}(T)\right) \subseteq f^{-1}(\ker(T))$ *for every subset T of Y*;
(e) *for each $x \in X$ and each open set V in Y containing $f(x)$, there exists a closed set U in X containing x such that $f(U) \subseteq V$;*
(f) *for each $x \in X$ and each closed set V in Y non-containing $f(x)$, there exists an open set W in X non-containing x such that $f^{-1}(V) \subseteq W$;*

we have that $(a) \Leftrightarrow (\mathrm{b}) \Leftrightarrow (\mathrm{c}) \Leftrightarrow (\mathrm{d})$, $(\mathrm{a}) \Rightarrow (\mathrm{e}) \Leftrightarrow (\mathrm{f})$ *and* $(\mathrm{b}) \Rightarrow (\mathrm{f})$.

Proof: $(a) \Leftrightarrow$ (b) : Let V be an open set in Y. Then $Y\setminus V$ is a closed set in Y. By hypothesis, $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is an open set in X. We have $f^{-1}(V)$ is a closed set in X and therefore f is contra-continuous. The converse can be obtained similarly.

(b) $\Rightarrow$ (c) : Let S be any subset of X. Suppose that $y \notin ker(f(S))$. Then by Lemma 3.2, there exists a closed set F in X containing y such that $f(S) \cap F = \emptyset$. Thus, we have $\mathrm{cl}(S) \cap f^{-1}(F) = \emptyset$ since $f^{-1}(F)$ is open in X. Since $y \in F$, then $f^{-1}(y) \in f^{-1}(F)$ and hence $f^{-1}(F) \notin \mathrm{cl}(S)$. Therefore $y \notin f(\mathrm{cl}(S))$. This implies that $f(\mathrm{cl}(S)) \subseteq \ker(f(S))$.

(c) $\Rightarrow$ (d) : Let T be any subset of Y. By (e) and Lemma 3.2, we have $f(\mathrm{cl}(f^{-1}(T)) \subseteq \ker\left(f\left(f^{-1}(T)\right)\right) \subseteq \ker(\mathrm{T})$ and $\mathrm{cl}(f^{-1}(T)) \subseteq f^{-1}\left(f\left(\mathrm{cl}\left(f^{-1}(T)\right)\right)\right) \subseteq f^{-1}(\ker(T))$.

(d) $\Rightarrow$ (a) : Let V be any open set in Y. Then by Lemma 3.2, we have $\mathrm{cl}\left(f^{-1}(V)\right) \subseteq f^{-1}(\ker(V)) = f^{-1}(V)$ since V is open in Y and thus $\mathrm{cl}\left(f^{-1}(V)\right) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is closed in X.

(a) $\Rightarrow$ (e) : Let V be an open set in Y containing $f(x)$. Then $x \in f^{-1}(V)$, and by (a), $f^{-1}(V)$ is closed set in X. By taking $U = f^{-1}(V)$, we have $x \in f^{-1}(V) = U$ and $f(U) = f(f^{-1}(V)) \subseteq V$.

$(e) \Leftrightarrow$ (f) : Let V be a closed set in Y not containing $f(x)$. Then $Y \setminus V$ is open set in Y containing $f(x)$, and by (e), there exists a closed set U in X containing x such that $f(U) \subseteq Y \setminus V$. Hence $X\setminus U$ is an open set in X not containing x such that $U \subseteq f^{-1}\left(f(U)\right) \subseteq f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Therefore $f^{-1}(V) \subseteq X \setminus U$. By taking $W = X \setminus U$, we have $f^{-1}(V) \subseteq W$.

Conversely, let V be an open set in Y containing $f(x)$. Then $Y \setminus V$ is a closed in Y not containing $f(x)$, and by (f), there exists an open set W in X containing x such that $f^{-1}(Y \setminus V) \subseteq W$. Hence $X \setminus W$ is a closed set in X containing x such that $X \setminus W \subseteq X \setminus f^{-1}(Y \setminus V) = f^{-1}(V)$. Therefore, $f(X \setminus W) \subseteq f(f^{-1}(V)) \subseteq V$. By taking $U = X \setminus W$, we have $f(U) \subseteq V$.

(b) $\Rightarrow$ (f) : Let V be a closed set in Y not containing $f(x)$. Then $x \notin f^{-1}(V)$, and by (b), $f^{-1}(V)$ is open set in X. By taking $W = f^{-1}(V)$, we have $f^{-1}(V) \subseteq W$. □

In Theorem 3.3, condition (e) does not imply condition (a) and condition (f) does not imply condition (b) as the following counterexamples show.

Example 3.4. *Let* $\mathbb{R}$ *be a set of real numbers with topology is cofinite topology and topology is discrete topology. If* B *is the collection of all singleton subsets of* $\mathbb{R}$*, then* B *is a* σ*-open base or the topology on* $\mathbb{R}$*. Let* $f:(\mathbb{R},\tau)\to(\mathbb{R},\sigma)$ *be an identity function. Then f is satisfied the condition* (e) *but not* (a) *since there exists a* σ*-open set* $(1,5]$ *in* $(\mathbb{R},\sigma)$ *such that* $f\big((1,5]\big)=(1,5]$ *is not* τ *-closed set in* $(\mathbb{R},\tau)$.

Example 3.5. *Let* $\mathbb{R}$ *be the set of real numbers with topology be the co-countable topology and be the discrete topology. Let* $f:(\mathbb{R},\tau)\to(\mathbb{R},\sigma)$ *be an identity function. Then f is satisfy the condition* (f) *but not* (b) *since there exists a* σ*-closed set* $[2,4]$ *in* $(\mathbb{R},\sigma)$*such that* $f([2,4])=[2,4]$ *is not* τ *-open in* $(\mathbb{R},\tau)$.

Theorem 3.6. *Let* $f:X\to Y$ *and* $g:Y\to Z$ *be two functions. Then* $g\circ f$ *is contra-continuous, if g is continuous and f is contra-continuous.*

Proof. Let W be an open set in Z. Since g is continuous, then $g^{-1}(W)$ is open in Y. Hence $f^{-1}\big(g^{-1}(W)\big)$ is closed in X because f is contra-continuous. But $f^{-1}\big(g^{-1}(W)\big)=(g\circ f)^{-1}(W)$. So, the composition function $g\circ f$ is contra-continuous. □

The concept of semi-open set and semi-closed set has been defined by using the concept of closure and interior operator in Definition 2.1 and it is also equivalent with the following definition.

Definition 3.7. [10] *A subset S of X is called*

(a) *semi-open set if* $O\subseteq S\subseteq \mathrm{cl}(O)$ *where* $O\in\tau$.

(b) *semi-closed set if* $\mathrm{int}(V)\subseteq S\subseteq V$ *where* $X\setminus V\in\tau$.

Theorem 3.8. *If S is a semi-open set and* $S \subseteq B \subseteq \mathrm{cl}(S)$, *then B is semi-open.*

Proof. Let S is semi-open set, then there exist an open set O such that $O \subseteq S \subseteq \mathrm{cl}(O)$. Because of $S \subseteq B \subseteq \mathrm{cl}(S)$, then $O \subseteq B$ since $O \subseteq S \subseteq B$. But $\mathrm{cl}(S) \subseteq \mathrm{cl}(O)$ because of $S \subseteq \mathrm{cl}(O)$ and $\mathrm{cl}(O)$ is closed set. So, $B \subseteq \mathrm{cl}(O)$. Thus we have $O \subseteq B \subseteq \mathrm{cl}(O)$, and therefore *B* is semi-open. □

Theorem 3.9. *Let* $f: X \to Y$ *be a mapping and let V be an arbitrary open set in Y. Then f is semi-continuous if and only if* $f^{-1}(V) \subseteq \mathrm{cl}\left(\mathrm{int}\left(f^{-1}(\mathrm{V})\right)\right)$.

Proof. Let *V* be an arbitrary open set in *Y*. Since *f* is semi-continuous, then $f^{-1}(V)$ is semi open in *X*. By Definition 2.1, $f^{-1}(V) \subseteq \mathrm{cl}\left(\mathrm{int}\left(f^{-1}(\mathrm{V})\right)\right)$. Conversely, let *V* be an arbitrary open set in *Y*. Since $f^{-1}(V) \subseteq \mathrm{cl}\left(\mathrm{int}\left(f^{-1}(\mathrm{V})\right)\right)$, then $f^{-1}(V)$ is semi-open in *X* by Definition 2.1. Hence *f* is semi-continuous. □

Theorem 3.10. *Let* $f: X \to Y$ *be a mapping and let V be an arbitrary open set in Y. Then f is precontinuous if and only if* $f^{-1}(V) \subseteq \mathrm{int}\left(\mathrm{cl}\left(f^{-1}(V)\right)\right)$

Proof. Let *V* be an arbitrary open set in *Y*. Since *f* is precontinuous, then $f^{-1}(V)$ is preopen in *X*. By Definition 2.1, $f^{-1}(V) \subseteq \mathrm{int}\left(\mathrm{cl}\left(f^{-1}(V)\right)\right)$. Conversely, let *V* be an arbitrary open set in *Y*. Since $f^{-1}(V) \subseteq \mathrm{int}\left(\mathrm{cl}\left(f^{-1}(V)\right)\right)$, then $f^{-1}(V)$ is pre-open in *X*. Hence *f* is precontinuous. □

Theorem 3.11. *Every closed set in X is* α *-closed.*

Proof: Let S be a closed set in *X*. Then $\mathrm{cl}(S) = S$. Therefore $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(S))) \subseteq \mathrm{cl}(S) = S$. Hence *S* is α -closed in *X*. □

Theorem 3.12. *Let $f: X \to Y$ be a mapping and let V be an arbitrary open set in Y. Then f is α-continuous if and only if* $f^{-1}(V) \subseteq \operatorname{int}\left(\operatorname{cl}\left(\operatorname{int}\left(f^{-1}(V)\right)\right)\right)$.

Proof. Let V be an arbitrary open set in Y. Since f is α-continuous, then $f^{-1}(V)$ is α-set in X. By Definition 2.1, $f^{-1}(V) \subseteq \operatorname{int}\left(\operatorname{cl}\left(\operatorname{int}\left(f^{-1}(V)\right)\right)\right)$. Conversely, let V be an arbitrary open set in Y. Since $f^{-1}(V) \subseteq \operatorname{int}\left(\operatorname{cl}\left(\operatorname{int}\left(f^{-1}(V)\right)\right)\right)$, then $f^{-1}(V)$ is α-set in X. Hence f is α-continuous.

Lemma 3.13. *Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of α-sets in topological space X. Then $\bigcup_{\alpha\in\Delta} A_\alpha$ is α-closed set in X.*

Proof. For each $\alpha \in \Delta$, since A_α is α-set in X, we have $A_\alpha \subseteq \operatorname{int}\left(\operatorname{cl}\left(\operatorname{int}(A_\alpha)\right)\right)$. Then $\bigcup_{\alpha\in\Delta} A_\alpha \subseteq \bigcup_{\alpha\in\Delta} \operatorname{int}\left(\operatorname{cl}\left(\operatorname{int}(A_\alpha)\right)\right) \subseteq \operatorname{int}(\bigcup_{\alpha\in\Delta} \operatorname{cl}(\operatorname{int}(A_\alpha))) \subseteq \operatorname{int}\left(\operatorname{cl}(\bigcup_{\alpha\in\Delta} \operatorname{int}(A_\alpha))\right) \subseteq \operatorname{int}\left(\operatorname{cl}\left(\operatorname{int}(\bigcup_{\alpha\in\Delta} A_\alpha)\right)\right)$. Therefore, $\bigcup_{\alpha\in\Delta} A_\alpha$ is α-set in X. □

Theorem 3.14. *Let $f: X \to Y$ be a mapping, then the following statements are equivalent.*

(a) *f is α-continuous*

(b) *for each $x \in X$ and each open set V in Y containing $f(x)$ there exists $W \in \alpha(X)$ such that $x \in W$ and $f(W) \subseteq V$*

(c) *$f^{-1}(V)$ is an α-closed in X for every closed set V in Y.*

(d) *$f(\operatorname{cl}\left(\operatorname{int}\left(\operatorname{cl}(S)\right)\right)) \subseteq \operatorname{cl}(f(S))$ for each $S \subseteq X$.*

(e) *$\operatorname{cl}\left(\operatorname{int}\left(\operatorname{cl}\left(f^{-1}(T)\right)\right)\right) \subseteq f^{-1}\left(\operatorname{cl}(T)\right)$ for each $T \subseteq Y$.*

Proof. (a) $\Leftrightarrow (b)$: Let V be an open set in Y containing $f(x)$. Then $x \in f^{-1}(V)$ and by (a), $f^{-1}(V)$ is α-set in X. By taking $W = f^{-1}(V)$, we have $x \in f^{-1}(V) = W$, and $f(W) = f\left(f^{-1}(V)\right) \subseteq V$. Conversely, let V be an open set in Y containing $f(x)$. By (a), there exists $W_x \in \alpha(X)$ such that $x \in W_x$ and $f(W_x) \subseteq V$. Then $x \in W_x \subseteq f^{-1}(V)$ and $f^{-1}(V) = \bigcup_{x\in f^{-1}(V)} W_x$. This shows that $f^{-1}(V) \in \alpha(X)$ by Lemma 3.13.

(a) $\Leftrightarrow (c)$: Let V a closed set in Y. Then $Y\backslash V$ is an open set in Y. By (a), $f^{-1}(Y\backslash V) \in \alpha(X)$. But $f^{-1}(Y\backslash V) = X\backslash f^{-1}(V) \in \alpha(X)$. Thus we have $f^{-1}(V)$ is α-closed in X. The converse can be obtained similarly.

(c) $\Rightarrow (d)$: Let S be an arbitrary subset of X. Then $\mathrm{cl}\big(f(S)\big)$ is a closed set in Y. By (c), $f^{-1}\Big(\mathrm{cl}\big(f(S)\big)\Big)$ is an α-closed in X. So, $\mathrm{cl}\bigg(\mathrm{int}\Big(f^{-1}\Big(\mathrm{cl}\big(f(S)\big)\Big)\Big)\bigg) \subseteq f^{-1}\Big(\mathrm{cl}\big(f(S)\big)\Big)$. Therefore

$$\mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(S)\big)\Big) \subseteq \mathrm{cl}\Big(\mathrm{int}\Big(f^{-1}\big(f(S)\big)\Big)\Big) \subseteq \mathrm{cl}\bigg(\mathrm{int}\Big(f^{-1}\Big(\mathrm{cl}\big(f(S)\big)\Big)\Big)\bigg) \subseteq f^{-1}\Big(\mathrm{cl}\big(f(S)\big)\Big) \quad .$$

This implies $f\bigg(\mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(S)\big)\Big)\bigg) \subseteq \mathrm{cl}\big(f(S)\big)$.

(d) $\Rightarrow (e)$: Let T be an arbitrary set in Y and let $S = f^{-1}(T)$. Then, by (d), we have $f\bigg(\mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(S)\big)\Big)\bigg) \subseteq \mathrm{cl}\big(f(S)\big) \subseteq \mathrm{cl}\Big(f\big(f^{-1}(T)\big)\Big) \subseteq \mathrm{cl}(T)$. This implies that $\mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(S)\big)\Big) \subseteq f^{-1}\big(\mathrm{cl}(T)\big)$. That is $\mathrm{cl}\bigg(\mathrm{int}\Big(\mathrm{cl}\big(f^{-1}(T)\big)\Big)\bigg) \subseteq f^{-1}\big(\mathrm{cl}(T)\big)$.

(e) $\Rightarrow$ (c) : Let V be an arbitrary closed set in Y. According to (e), $\mathrm{cl}\bigg(\mathrm{int}\Big(\mathrm{cl}\big(f^{-1}(V)\big)\Big)\bigg) \subseteq f^{-1}\big(\mathrm{cl}(V)\big)$. Since V is closed, $\mathrm{cl}(V) = V$ and hence $\mathrm{cl}\bigg(\mathrm{int}\Big(\mathrm{cl}\big(f^{-1}(V)\big)\Big)\bigg) \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is α-closed in X. □

Corollary 3.15. *Let $f: X \to Y$ be α-continuous, then;*

(a) $f\big(\mathrm{cl}(S)\big) \subseteq \mathrm{cl}\big(f(S)\big)$ *for each $S \in PO(X,\tau)$.*

(b) $\mathrm{cl}\big(f^{-1}(A)\big) \subseteq f^{-1}\big(\mathrm{cl}(A)\big)$ *for each open set A in Y.*

Proof. (a) Let S be any arbitrary preopen set in (X,τ). Then $S \subseteq \mathrm{int}\big(\mathrm{cl}(S)\big)$ so that$\mathrm{cl}(S) \subseteq \mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(S)\big)\Big)$. Since it is always true that $\mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(S)\big)\Big) \subseteq \mathrm{cl}(S)$, we will have $\mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(S)\big)\Big) = \mathrm{cl}(S)$. Therefore, $f\big(\mathrm{cl}(S)\big) = f\bigg(\mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(S)\big)\Big)\bigg) \subseteq \mathrm{cl}\big(f(S)\big)$ by part (d) of Theorem 3.14.

(b) Let A be any arbitrary open set in Y. Since f is α-continuous, then $f^{-1}(A)$ is an α-set in X. Hence, $f^{-1}(A) \subseteq \mathrm{int}\bigg(\mathrm{cl}\Big(\mathrm{int}\big(f^{-1}(A)\big)\Big)\bigg) \subseteq \mathrm{int}\Big(\mathrm{cl}\big(f^{-1}(A)\big)\Big)$. This implies

$\mathrm{cl}(f^{-1}(A)) \subseteq \mathrm{cl}\left(\mathrm{int}\left(\mathrm{cl}(f^{-1}(A))\right)\right)$. But since $A \subseteq Y$, we have $\mathrm{cl}\left(\mathrm{int}\left(\mathrm{cl}(f^{-1}(A))\right)\right) \subseteq f^{-1}(\mathrm{cl}(A))$ by part (e) of Theorem 3.14. Therefore, $\mathrm{cl}(f^{-1}(A)) \subseteq f^{-1}(\mathrm{cl}(A))$ □

Lemma 3.16. *A subset S of X is locally closed if and only if $S = U \cap \mathrm{cl}(S)$ for some open set U of X.*
Proof. Necessity. Let S be a locally closed subset of X. Then $S = U \cap S$ where U is open and F is closed in X. We have $\mathrm{cl}(S) \subseteq F$, hence $S \subseteq U \cap \mathrm{cl}(S) \subseteq U \cap F = S$. This shows that $S = U \cap \mathrm{cl}(S)$.
Sufficiency. Let $S = U \cap \mathrm{cl}(S)$ where U is open in X and $\mathrm{cl}(S)$ is closed in X, then S is locally closed in X. □

Note that, $x \in \mathrm{cl}(A)$ if and only if every open set *U* containing *x* intersects *A*, see [13].

Lemma 3.17. *Let A be an open set in topological space X. Then $A \cap \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cap B)$ for every $B \subseteq X$.*
Proof. Suppose that $x \in A \cap \mathrm{cl}(B)$. Then $x \in A$ and $x \in \mathrm{cl}(B)$, so $x \in A$ and $O \cap B \neq \emptyset$ for every open set *O* containing *x*. Since *A* is also open set containing *x*, then $O \cap A$ is open set containing *x*. Hence $(O \cap A) \cap B \neq \emptyset$, i.e., $O \cap (A \cap B) \neq \emptyset$, for every open set *O* containing *x*. Thus $x \in \mathrm{cl}(A \cap B)$. □

Theorem 3.18. *Let S be a subset of a topological space X. Then S is an A-set if and only if S is semi-open and locally closed*.
Proof. Let $S \in A(X, \tau)$, so $S = U \cap F$ where $U \in \tau$ and $F \in RC(X, \tau)$. Obviously that *S* is locally closed since F is also closed set in X. Now $\mathrm{int}(S) = U \cap \mathrm{int}(F)$ and $S = U \cap \mathrm{cl}(\mathrm{int}(F)) \subseteq \mathrm{cl}(U \cap \mathrm{int}(F)) = \mathrm{cl}(\mathrm{int}(S))$ by Lemma 3.17. Hence *S* is semi-open.
Conversely, let S be semi-open and locally closed, so that $S \subseteq \mathrm{cl}(\mathrm{int}(S))$ and $S = U \cap \mathrm{cl}(\mathrm{int}(S))$ by Lemma 3.16. Then $\mathrm{cl}(S) \subseteq \mathrm{cl}\left(\mathrm{cl}(\mathrm{int}(S))\right) = \mathrm{cl}(\mathrm{int}(S))$. Therefore $\mathrm{cl}(S)$ is a regular closed set. So, *S* is an *A*-set. □

Theorem 3.19. *A function* $f: X \to Y$ *is almost continuous if and only if* $f^{-1}(V) \subseteq \operatorname{int}\left(f^{-1}\left(\operatorname{int}(\operatorname{cl}(V))\right)\right)$ *for any open set V in Y.*

Proof. Necessity. Let V be an arbitrary open set in Y and let $x \in f^{-1}(V)$ then $f(x) \in V$. Since V is open, it is a neighborhood of $f(x)$ in Y. Since f is almost continuous at point x, there exists an open neighborhood U of x in X such that $f(U) \subseteq \operatorname{int}(\operatorname{cl}(V))$. This implies that $U \subseteq f^{-1}\left(\operatorname{int}(\operatorname{cl}(V))\right)$, thus $x \in U \subseteq f^{-1}\left(\operatorname{int}(\operatorname{cl}(U))\right)$. Thus, $f^{-1}(V) \subseteq \operatorname{int}\left(f^{-1}\left(\operatorname{int}(\operatorname{cl}(V))\right)\right)$.

Sufficiency. Let V be an arbitrary open set in Y such that $f(x) \in V$. Then, $x \in f^{-1}(V) \subseteq \operatorname{int}\left(f^{-1}\left(\operatorname{int}(\operatorname{cl}(V))\right)\right)$ .Take $U = \operatorname{int}\left(f^{-1}\left(\operatorname{int}(\operatorname{cl}(V))\right)\right)$, then $f(U) \subseteq f\left(f^{-1}\left(\operatorname{int}(\operatorname{cl}(V))\right)\right) \subseteq \operatorname{int}(\operatorname{cl}(V))$ such that $f(U) = \operatorname{int}(\operatorname{cl}(V))$. By Definition 2.3, f is almost continuous. □

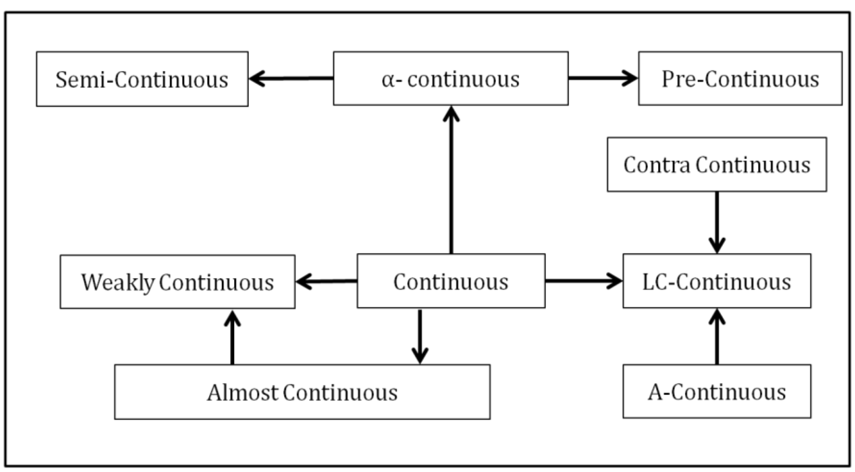
Theorem 3.20. *A function* $f: X \to Y$ *is weakly continuous if only if* $f^{-1}(V) \subseteq \operatorname{int}\left(f^{-1}(\operatorname{cl}(V))\right)$ *for each open subset V of Y.*

Proof. Necessity. Let V be an arbitrary open set in Y and let $x \in f^{-1}(V)$ then $f(x) \in V$. Then, there exists an open set U in X such that $x \in U$ and $f(U) \subseteq \operatorname{cl}(V)$. Hence, $x \in U \subseteq f^{-1}(\operatorname{cl}(V))$ and $x \in \operatorname{int}\left(f^{-1}(\operatorname{cl}(V))\right)$ since U is open.

Sufficiency. Let $x \in X$ and $f(x) \in V$. Then, $x \in f^{-1}(V) \subseteq \operatorname{int}\left(f^{-1}(\operatorname{cl}(V))\right)$. Let $U = \operatorname{int}\left(f^{-1}(\operatorname{cl}(V))\right)$ then U is open set containing x and $f(U) = f\left(\operatorname{int}\left(f^{-1}(\operatorname{cl}(V))\right)\right) \subseteq f\left(f^{-1}(\operatorname{cl}(V))\right) \subseteq \operatorname{cl}(V)$. Hence, f is weakly continuous. □

DECOMPOSITION OF CONTINUITY

Observe that, continuous function and contra continuous functions are independent concepts. Refer to the discussion below, Proposition 4.1 showed that continuity implies almost continuity but the converse is not true as shown in Example 4.2. Every continuous function is also a weakly continuous as shown in Proposition 4.3 but the converse is not true as shown in Example 4.4. Moreover, Proposition 4.5 showed that almost continuity implies weakly continuity but the converse is not true in general as shown in Example 4.6. By the above definitions, it is very clear that every continuous function is *LC*-continuous function and every continuous function is α -continuous but the converses are not true as being shown in Example 4.7 and Example 4.8. Proposition 4.9 showed that α -continuity implies semi-continuity and pre-continuity but the converses are not true as being shown in Example 4.10 and Example 4.11. Furthermore, contra continuous function implies *LC*-continuous but the converse is not true. See [4]. LC-continuous function does not imply *A*-continuous function as being shown in [6]. The following diagram summarizes the above discussions.



Proposition 4.1. *If a function $f: X \to Y$ is continuous, then f is almost continuous.*
Proof. Let V be an open set in Y, then $V \subseteq \operatorname{int}(\operatorname{cl}(V))$. Since f is continuous, $f^{-1}(V)$ is open in X such that $f^{-1}(V) \subseteq f^{-1}\left(\operatorname{int}(\operatorname{cl}(V))\right)$. Since $f^{-1}(V) = \operatorname{int}(f^{-1}(V))$ in X, $f^{-1}(V) = \operatorname{int}(f^{-1}(V)) \subseteq \operatorname{int}\left(f^{-1}\left(\operatorname{int}(\operatorname{cl}(V))\right)\right)$. By Theorem 3.19, f is almost continuous. □

Example 4.2. [14] *Let ℝ be the set of real numbers with indiscrete topology $\tau = \{\emptyset, \mathbb{R}\}$, and let $Y = \{a, b\}$ with topology $\sigma = \{\emptyset, Y, \{a\}\}$. Let $f: (\mathbb{R}, \tau) \to (Y, \sigma)$ be defined as follows*

$$f(x) = \begin{cases} a & if\ x\ is\ rational \\ b & if\ x\ is\ irrational \end{cases}$$

Then f is almost continuous but f is not continuous because $f^{-1}(\{a\})$ is not open in $(\mathbb{R}, \tau)$ for $\{a\} \in \sigma$.

Proposition 4.3. *If a function $f: X \to Y$ is continuous then f is weakly continuous*.
Proof. Let V be an open set in Y. Since f is continuous, there exists an open set U in X such that $f(U) \subseteq V$. It follows that $f(U) \subseteq \operatorname{cl}(V)$ and by Definition 2.3, f is weakly continuous. □

Example 4.4. *Let $X = \{a, b, c\}$ have topologies $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{b\}\}$ and $\sigma = \{\emptyset, X, \{a, c\}, \{b, c\}, \{c\}\}$ and let $f: (X, \tau) \to (X, \sigma)$ be identity function. Then f is weakly continuous but f is not continuous since $f^{-1}(\{a, c\}) = \{a, c\}$ is not open in (X, τ) for $\{a, c\}$ is open in (X, σ).*

Proposition 4.5. *If $f: X \to Y$ is almost continuous function then f is weakly continuous.*
Proof. Let V be an arbitrary open set in Y. Since f is almost continuous, there exists an open set U in X such that $f(U) \subseteq \operatorname{int}(\operatorname{cl}(V))$. So we have $f(U) \subseteq \operatorname{cl}(V)$ and obviously that, f is weakly continuous. □

Example 4.6. *Let* $X = \{x, y, z\}$ *and* $\tau = \{\emptyset, \{x, y\}, \{y\}, X\}$. *Let* $Y = \{a, b, c\}$ *and* $\sigma = \{\emptyset, \{a\}, \{c\}, \{a, c\}, Y\}$. *Now, let* $f: X \to Y$ *be a function defined as follows:* $f(x) = f(z) = a, f(y) = b$. *Hence f is weakly continuous but not almost continuous since* $f^{-1}(\{a\}) = \{x, z\} \nsubseteq int\left(f^{-1}\left(int\big(cl(\{a\})\big)\right)\right) = \emptyset$ *for* $\{a\}$ *is open in* (Y, σ).

Example 4.7. *Define a real-valued function* $f: \mathbb{R} \to \mathbb{R}$ *together with usual topology by setting:*

$$f(x) = \begin{cases} 1 & if\ x > 0 \\ -1 & if\ x < 0 \end{cases}$$

For any subset V of $\mathbb{R}$, *we have* $f^{-1}(V) = V \cap (-\infty, 0]$ *if* $1 \notin V$ *and* $f^{-1}(V) = V \cup (0, \infty)$ *if* $1 \in V$. *Then f is LC-continuous but not continuous at point* 0.

Example 4.8. *Let* $X = \{a, b, c, d\}$ *with topology* $\tau = \{\emptyset, \{a, b\}, \{b, c, d\}, \{b\}, X\}$ *and let* $Y = \{x, y, z\}$ *with topology* $\sigma = \{\emptyset, \{x, z\}, \{z\}, Y\}$. *Let* $f: (X, \tau) \to (Y, \sigma)$ *be a function defined as follows:* $f(a) = f(d) = x, f(c) = y$. *Then f is* α *-continuous but not continuous since* $f^{-1}(\{x, z\}) = \{a, b, d\}$ *is not open in* (X, τ) *while* $\{x, z\}$ *is open in* (Y, σ).

Proposition 4.9. *If a function* $f: X \to Y$ *is* α*-continuous, then f is semi-continuous and pre-continuous.*

Proof. Let *V* be an open set in *Y*. Since *f* is α -continuous function, then $f^{-1}(V)$ is α -set in *X*, i.e., $f^{-1}(V) \subseteq \mathrm{int}\left(\mathrm{cl}\left(\mathrm{int}\big(f^{-1}(V)\big)\right)\right) \subseteq \mathrm{cl}\left(\mathrm{int}\big(f^{-1}(V)\big)\right)$. So $f^{-1}(V)$ is semi-open in *X* and hence *f* is semi-continuous. Also $f^{-1}(V) \subseteq \mathrm{int}\left(\mathrm{cl}\left(\mathrm{int}\big(f^{-1}(V)\big)\right)\right) \subseteq \mathrm{int}\left(\mathrm{cl}\big(f^{-1}(V)\big)\right)$, so $f^{-1}(V)$ is pre-open in *X*. Therefore *f* is pre-continuous. □

Example 4.10. *Let* $X = \{a, b, c\}$ *with topologies* $\tau = \{\emptyset, \{a, c\}, \{a\}, \{c\}, X\}$ *and* $\sigma = \{\emptyset, \{a, b\}, \{b, c\}, \{b\}, \{a, c\}, \{c\}, \{a\}, X\}$. *Let* $f: (X, \tau) \to (X, \sigma)$ *be the function defined as follows:* $f(a) = a, f(c) = f(b) = b$. *Then f is semi-continuous but not* α *-continuous since* $f^{-1}(\{b, c\}) = \{b, c\} \nsubseteq int\left(cl\left(int\big(f^{-1}(\{b, c\})\big)\right)\right) = \{c\}$ *by Theorem* 3.12.

Example 4.11. *Let* $X = \{a, b, c\}$ *with indiscrete topology and* $Y = \{x, y, z\}$ *with topology* $\sigma = \{\emptyset, \{x, y\}, \{y, z\}, \{y\}, \{x, z\}, \{x\}, \{z\}, Y\}$. *Let* $f: (X, \tau) \to (Y, \sigma)$ *be an injective mapping defined as follows:* $f(a) = x, f(b) = y, f(c) = z$. *Then* f *is pre-continuous, but not* α*-continuous since* $cl\left(int\left(cl\left(f^{-1}(\{x\})\right)\right)\right) = X \nsubseteq f^{-1}\left(cl(\{x\})\right) = \{a\}$ *by Theorem* 3.14.

CONCLUSION

As a result, the connection and relationship of some generalized continuities has been characterized and illustrated by some counterexamples. The relationship between some generalized continuities has been found to be useful in the study of generalized continuities in topological spaces.

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