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Pairwise Nearly Regular-Lindelöf Bitopological Spaces

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ABSTRACT

In this paper, we shall introduce and study the pairwise nearly regular-Lindelöf bitopological spaces and investigate some of their characterizations. Moreover we study the relationships between i-Lindelöf, (i, j)-nearly Lindelöf, (i, j)-almost Lindelöf, (i, j)-almost regular-Lindelöf and (i, j)-nearly regular-Lindelöf spaces.

Keywords: Bitopological space, (i, j)-nearly regular-Lindelöf, pairwise nearly regular-Lindelöf, (i, j)-regular open, (i, j)-regular cover.

INTRODUCTION

The study of bitopological spaces was first initiated by Kelly (1963) and thereafter a large number of papers have been done to generalize the topological concepts to bitopological setting. In literature there are several generalizations of the notion of Lindelöf spaces and these are studied separately for different reasons and purposes. Balasubramaniam (1982) introduced and studied the notion of nearly Lindelöf spaces and Cammaroto and Santoro (1996) studied and gave further new results about these spaces followed by Kılıçman and Fawakhreh (2000). In the same paper Cammaroto and Santoro introduced the notion of nearly regular-Lindelöf spaces by using regular covers and leave open the study of this new concept. Kılıçman and Fawakhreh (2001) studied this new generalization of Lindelöf spaces and obtained some results.

Recently, Salleh and Kılıçman (2009) introduced and studied the notion of pairwise nearly Lindelöf spaces in bitopological setting and extended some results due to Cammaroto and Santoro (1996) and Kılıçman and Fawakhreh (2000). The purpose of this paper is to define the notion of nearly regular-Lindelöf property in bitopological spaces, which we will call pairwise nearly regular-Lindelöf spaces and investigate some of their characterizations.

In section 3, we shall introduce the concept of pairwise nearly regular-Lindelöf bitopological spaces by using (i, j)-regular cover. This study begin by investigating the (i, j)-nearly regular-Lindelöf property and some results obtained. Furthermore, we study the relationships between i-Lindelöf, (i, j)-nearly Lindelöf, (i, j)-almost Lindelöf, (i, j)-almost regular-Lindelöf and (i, j)-nearly regular-Lindelöf spaces.

PRELIMIARIES

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) (or simply X) are always mean topological spaces and bitopological spaces, respectively unless explicitly stated. In this paper, if P is a topological property, then (τ_i, τ_j)-P denotes an analogue of this property for τ_i has property P with respect to τ_j, and p-P denotes the conjunction (τ_1, τ_2)-P $\wedge$ (τ_2, τ_1)-P, i.e., p-P denotes an absolute bitopological analogue of P. As we shall see below, sometimes (τ_1, τ_2)-P $\Leftrightarrow$ (τ_2, τ_1)-P (and thus $\Leftrightarrow$ p-P) so that it suffices to consider one of these three bitopological analogue. Also sometimes τ_1-P $\Leftrightarrow$ τ_2-P and thus P $\Leftrightarrow$ τ_1-P $\wedge$ τ_2-P, i.e., (X, τ_i) has property P for each i = 1, 2.

Also note that (X, τ_i) has a property P $\Leftrightarrow$ (X, τ_1, τ_2) has a property τ_i-P. The prefixes (τ_i, τ_j)- or τ_i- will be replaced by (i, j)- or i-, respectively, if there is no chance for confusion. In this paper always i, j $\in$ {1, 2} and i $\neq$ j. By i-open cover of X, we mean that the cover of X by i-open sets in X; similar for the (i, j)-regular open cover of X etc. By i-int(A) and i-cl(A), we shall mean the interior and the closure of a subset A of X with respect to topology τ_i, respectively. The reader may consult [2] for the detail notations.

Definition 2.1. [7, 16] A subset S of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-regular open (resp. (i, j)-regular closed) if i-int(j-cl(S)) = S (resp. i-cl(j-int(S)) = S). S is called pairwise regular open (resp. pairwise regular closed) if it is both (1, 2)-regular open and (2, 1)-regular open (resp. (1, 2)-regular closed and (2, 1)-regular closed).

Definition 2.2. Let (X, τ_1, τ_2) be a bitopological space. A subset F of X is said to be
(i) i-open if F is open with respect to τ_i in X, F is called open in X if it is both 1-open and 2-open in X, or equivalently, $F \in U$ for $U \subseteq (\tau_1 \cap \tau_2)$ in X;
(ii) i-closed if F is closed with respect τ_i in X, F is called closed in X if it is both 1-closed and 2-closed in X, or equivalently, $X \setminus F \in (\tau_1 \cap \tau_2)$ in X.

Definition 2.3. [5, 10] A bitopological space (X, τ_1, τ_2) is said to be i-Lindelöf if the topological space (X, τ_i) is Lindelöf. X is called Lindelöf if it is i-Lindelöf for each i = 1, 2.

Definition 2.4. [6, 7] A bitopological space (X, τ_1, τ_2) is said to be (i, j)-regular if for each point

$x \in X$ and for each i-open set V of X containing x, there exists an i-open set U such that $x \in U \subseteq$ j-cl(U) $\subseteq$ V. X is called pairwise regular if it is both (1, 2)-regular and (2, 1)-regular.

Definition 2.5. [7, 17] A bitopological space X is said to be (i, j)-almost regular if for each $x \in X$ and for each (i, j)-regular open set V of X containing x, there is an (i, j)-regular open set U such that $x \in U \subseteq$ j-cl(U) $\subseteq$ V. X is called pairwise almost regular if it is both (1, 2)-almost regular and (2, 1)-almost regular.

Definition 2.6. [7] A bitopological space X is said to be (i, j)-semiregular if for each $x \in X$ and for each i-open set V of X containing x, there is an i-open set U such that $x \in U \subseteq$ i-int(j-cl(U)) $\subseteq$ V. X is called pairwise semiregular if it is both (1, 2)-semiregular and (2, 1)-semiregular.Menemui Matematik .Vol.30.2013

Definition 2.7. [11, 12, 15] *A bitopological space X is said to be* (i, j)*-nearly Lindelöf* (*resp.* (i, j)*-almost Lindelöf*) *if for every i-open cover* $\{U_\alpha : \alpha \in \Delta\}$ *of X, there exists a countable subset* $\{\alpha_n : n \in \mathbb{N}\}$ *of* Δ *such that* $X = \bigcup_{n \in \mathbb{N}}$ *i*-int(*j*-cl(U_{α_n})) (*resp.* $X = \bigcup_{n \in \mathbb{N}}$ *j*-cl(U_{α_n})). *X is called pairwise nearly Lindelöf* (*resp. pairwise almost Lindelöf*) *if it is both* (1, 2)*-nearly Lindelöf* (*resp.* (1, 2)*-almost Lindelöf*) *and* (2, 1)*-nearly Lindelöf* (*resp.* (2, 1)*-almost Lindelöf*).

PAIRWISE NEARLY REGUAR-LINDELÖF SPACES

Definition 3.1. [14] An i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of a bitopological space X is said to be (i, j)-regular cover if for every $\alpha \in \Delta$, there exists a non-empty (j, i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha \in \Delta}$ i-int(C_α). $\{U_\alpha : \alpha \in \Delta\}$ is called pairwise regular cover if it is both (1, 2)-regular cover and (2, 1)-regular cover.

Definition 3.2. [14] A bitopological space X is said to be (i, j)-almost regular-Lindelöf if for every (i, j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n \in \mathbb{N}}$ j-cl(U_{α_n}). X is called pairwise almost regular-Lindelöf if it is both (1, 2)-almost regular-Lindelöf and (2, 1)-almost regular-Lindelöf.

The following definition extend the notion of nearly regular-Lindelöf spaces due to Cammaroto and Santoro [3] to bitopological setting.

Definition 3.3. A bitopological space X is said to be (i, j)-nearly regular-Lindelöf if for every (i, j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n \in \mathbb{N}}$ i-int(j-cl(U_{α_n}). X is called pairwise nearly regular-Lindelöf if it is both (1, 2)-nearly regular-Lindelöf and (2, 1)-nearly regular-Lindelöf.

From Definition 3.1, every U_α is consider i-open set in X. If for every $\alpha \in \Delta$, U_α is an (i,j)-regular open subset of X, then $\{U_\alpha : \alpha \in \Delta\}$ is called (i,j)-regular cover of X by (i,j)-regular open subsets of X. The following theorem is obvious from the definitions.

Theorem 3.1. A bitopological space X is (i,j)-nearly regular-Lindelöf if and only if every (i,j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X by (i,j)-regular open subsets of X has a countable subcover.

Proof. Necessity. Let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular cover of X by (i,j)-regular open subsets of X. Since (i,j)-regular open set is also i-open, then $\{U_\alpha : \alpha \in \Delta\}$ is an (i,j)-regular cover of X. Since X is (i,j)-nearly regular-Lindelöf, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that X = $\bigcup_{n\in\mathbb{N}}$ i-int(j-cl(U_{α_n}) = $\bigcup_{n\in\mathbb{N}} U_{\alpha_n}$.
Sufficiency. Let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular cover of X. Hence {i-int(j-cl(U_α) : $\alpha \in \Delta$} is an (i,j)-regular cover of X by (i,j)-regular open subsets of X. So there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that X = $\bigcup_{n\in\mathbb{N}}$ i-int(j-cl(U_{α_n}). This completes the proof. □

Obviously by the definitions, every (i,j)-nearly Lindelöf space is (i,j)-nearly regular-Lindelöf and every (i,j)-nearly regular-Lindelöf space is (i,j)-almost regular-Lindelöf.

Problem 3.1. Is (i,j) (i,j)-nearly regular-Lindelöf spaces imply (i,j)-nearly Lindelöf?

Problem 3.2. Is (i,j)-almost regular-Lindelöf spaces imply (i,j)-nearly regular-Lindelöf?

The authors conjecture that the answer of both problems are no. We can answer Problem 3.1 by some restrictions on the space as follows.

Proposition 3.1. An (i,j)-nearly regular-Lindelöf and (i,j)-almost regular space X is (i,j)-nearly Lindelöf.

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular open cover of X. For each $x \in X$, there exists $\alpha_x \in \Delta$ such that $x \in U_{\alpha_x}$. Since X is (i,j)-almost regular, there an (i,j)-regular open subsets V_{α_x} of X such that $x \in V_{\alpha_x} \subseteq$ j-cl(V_{α_x}) $\subseteq U_{\alpha_x}$. Since for each $\alpha_x \in \Delta$, there exists a (j,i)-regular closed set j-cl(V_{α_x}) in X such that j-cl(V_{α_x}) $\subseteq U_{\alpha_x}$ and X = $\bigcup_{x\in X} V_{\alpha_x}$ = $\bigcup_{x\in X}$ i-int(j-cl(V_{α_x})), the family $\{U_{\alpha_x} : x \in X\}$ is an (i,j)-regular cover of X by (i,j)-regular open subsets of X. Since X is (i,j)-nearly regular-Lindelöf, there exists a countable subset of points $x_1, \ldots, x_n, \ldots$ of X such that X = $\bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}}$. Therefore X is (i,j)-nearly Lindelöf. □

Corollary 3.1. Let X be an (i,j)-almost regular space. Then X is (i,j)-nearly regular-Lindelöf if and only if it is (i,j)-nearly Lindelöf.

Note that, if (X, τ_1, τ_2) is (i, j)-semiregular and (i, j)-nearly Lindelöf then it is i-Lindelöf [15], and every (i, j)-regular space is (i, j)-semiregular and (i, j)-almost regular [14]. Thus by these facts and Proposition 3.1, we conclude the following proposition.

Proposition 3.2. An (i, j)-regular and (i, j)-nearly regular-Lindelöf space X is i-Lindelöf.

Corollary 3.2. Let X be an (i, j)-regular space. Then X is (i, j)-nearly regular-Lindelöf if and only if it is i-Lindelöf.

Note that, if X is an (i, j)-almost regular and (i, j)-nearly Lindelöf space, then X is (i, j)-nearly paracompact [13]. Thus on using this fact and Proposition 3.1, we obtain the following proposition.

Proposition 3.3. An (i, j)-almost regular and (i, j)-nearly regular-Lindelöf space X is (i, j)-nearly paracompact.

Since (j, i)-extremally disconnected spaces is (i, j)-almost regular (see [11, 12]), it is easy to prove the following proposition by a direct consequence of Proposition 3.1.

Proposition 3.4. Let (X, τ_1, τ_2) be a (j, i)-extremally disconnected and (i, j)-nearly regular-Lindelöf space, then it is (i, j)-nearly Lindelöf.

Corollary 3.3. Let (X, τ_1, τ_2) be a (j,i)-extremally disconnected. Then X is (i, j)-nearly regular-Lindelöf if and only if it is (i, j)-nearly Lindelöf.

Theorem 3.2. *If X is (i, j)-almost Lindelöf space, then it is (i, j)-nearly regular-Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an (i, j)-regular cover of X. By Definition 3.1, for each $\alpha \in \Delta$ there exists a nonempty (j, i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha \in \Delta} i\text{-int}(C_\alpha)$. So, $\{i\text{-int}(C_\alpha) : \alpha \in \Delta\}$ forms an i-open cover of X refining $\{U_\alpha : \alpha \in \Delta\}$. Since X is (i, j)-almost Lindelöf, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n \in \mathbb{N}} j\text{-cl}(i\text{-int}(C_{\alpha_n})) = \bigcup_{n \in \mathbb{N}} C_{\alpha_n} \subseteq \bigcup_{n \in \mathbb{N}} U_{\alpha_n} \subseteq \bigcup_{n \in \mathbb{N}} i\text{-int}(j\text{-cl}(U_{\alpha_n}))$. This implies that X is (i, j)-nearly regular-Lindelöf. □

Corollary 3.4. If X is pairwise almost Lindelöf space, then it is pairwise nearly regular-Lindelöf.

The following theorem give some characterizations of (i, j)-nearly regular-Lindelöf spaces.

Theorem 3.3. Let X be a bitopological space. The following conditions are equivalent:

(i) X is (i,j)-nearly regular-Lindelöf ;

(ii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta}$ i-cl$(A_\alpha) = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n\in\mathbb{N}} C_{\alpha_n} = \emptyset$;

(iii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ satisfies $\bigcap_{n\in\mathbb{N}} C_{\alpha_n} \neq \emptyset$, the intersection $\bigcap_{\alpha\in\Delta}$ i-cl$(A_\alpha) \neq \emptyset$ for each (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$;

(iv) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta}$ i-cl$(A_\alpha) = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n\in\mathbb{N}}$ i-cl(j-int$(C_{\alpha_n})) = \emptyset$;

(v) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ satisfies $\bigcap_{n\in\mathbb{N}}$ i-cl(j-int$(C_{\alpha_n})) \neq \emptyset$, the intersection $\bigcap_{\alpha\in\Delta}$ i-cl$(A_\alpha) \neq \emptyset$ for each (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$.

Proof. (i) ⇔ (ii): Let $\{C_\alpha : \alpha \in \Delta\}$ be a family of (i,j)-regular closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta}$ i-cl$(A_\alpha) = \emptyset$. It follows that $X = X\setminus(\bigcap_{\alpha\in\Delta}$ i-cl$(A_\alpha)) = \bigcup_{\alpha\in\Delta}$ i-int$(X\setminus A_\alpha)$. Since $C_\alpha \subseteq A_\alpha =$ j-int(i-cl$(A_\alpha)) \subseteq$ i-cl(A_α), then i-int$(X\setminus A_\alpha) \subseteq X\setminus A_\alpha \subseteq X\setminus C_\alpha$. Therefore $X = \bigcup_{\alpha\in\Delta}$ i-int$(X\setminus A_\alpha) \subseteq \bigcup_{\alpha\in\Delta} (X\setminus C_\alpha)$. So, the family $\{X\setminus C_\alpha : \alpha \in \Delta\}$ is an (i,j)-regular cover of X by (i,j)-regular open subsets of X. Since every (i,j)-regular open set in X is also i-open, by (i) there exists a countable subfamily $\{X \setminus C_{\alpha_n} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}}$ i-int(j-cl$(X \setminus C_{\alpha_n})) = \bigcup_{n\in\mathbb{N}} (X \setminus C_{\alpha_n}) = X\setminus(\bigcap_{n\in\mathbb{N}} C_{\alpha_n})$. Therefore $\bigcap_{n\in\mathbb{N}} C_{\alpha_n} = \emptyset$.

Conversely, let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular cover of X. Then for each $\alpha \in \Delta$, there exists a (j,i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha\in\Delta}$ i-int(C_α). The family $\{X\setminus i$-int$(j$-cl$(U_\alpha)) : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X satisfies the conditions, for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset $X\setminus C_\alpha$ of X such that $X\setminus C_\alpha \supseteq X\setminus U_\alpha \supseteq X\setminus i$-int$(j$-cl$(U_\alpha))$ and $\bigcap_{\alpha\in\Delta}$ i-cl$(X\setminus C_\alpha) = X\setminus(\bigcup_{\alpha\in\Delta}$ i-int$(C_\alpha)) = X\setminus X = \emptyset$. By (ii), there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $\bigcap_{n\in\mathbb{N}} X\setminus i$-int$(j$-cl$(U_{\alpha_n})) = \emptyset$, i.e., $X\setminus(\bigcup_{n\in\mathbb{N}}$ i-int$(j$-cl$(U_{\alpha_n}))) = \emptyset$. Therefore $X = \bigcup_{n\in\mathbb{N}}$ i-int$(j$-cl$(U_{\alpha_n}))$ and (i) proved.

(i) ⇔ (iii): Let $\{C_\alpha : \alpha \in \Delta\}$ be a family of (i,j)-regular closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ satisfies $\bigcap_{n\in\mathbb{N}} C_{\alpha_n} \neq \emptyset$. Suppose that $\bigcap_{\alpha\in\Delta}$ i-cl$(A_\alpha) = \emptyset$ for each (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$. Hence $X = X\setminus\bigcap_{\alpha\in\Delta}$ i-cl$(A_\alpha) = \bigcup_{\alpha\in\Delta}$ i-int$(X\setminus A_\alpha)$. Since $C_\alpha \subseteq A_\alpha \subseteq$ i-cl(A_α), then i-int$(X\setminus A_\alpha) \subseteq X\setminus A_\alpha \subseteq X\setminus C_\alpha$. Therefore $X = \bigcup_{\alpha\in\Delta}$ i-int$(X\setminus A_\alpha) \subseteq \bigcup_{\alpha\in\Delta} (X\setminus C_\alpha)$. So, the family $\{X\setminus C_\alpha : \alpha \in \Delta\}$ is an (i,j)-regular cover of X by (i,j)-

regular open subsets of X. Since every (i, j)-regular open set in X is also i-open, by (i) there exists a countable subfamily $\{X \setminus C_{\alpha_n} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}}$ i-int(j-cl($X \setminus C_{\alpha_n}$)) $= \bigcup_{n\in\mathbb{N}}$ $(X \setminus C_{\alpha_n}) = X\setminus(\bigcap_{n\in\mathbb{N}} C_{\alpha_n})$. Therefore $\bigcap_{n\in\mathbb{N}} C_{\alpha_n} = \emptyset$ which is a contradiction.

Conversely, suppose that X is not (i, j)-nearly regular-Lindelöf. Then there exists an (i, j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X with no countable subfamily $\{U_{\alpha_n} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}}$ i-int(j-cl(U_{α_n})). Hence $X \neq \bigcup_{n\in\mathbb{N}}$ i-int(j-cl(U_{α_n})) for any countable subfamily $\{U_{\alpha_n} : n \in \mathbb{N}\}$. It follows that $X\setminus\bigcup_{n\in\mathbb{N}}$ i-int(j-cl(U_{α_n})) $\neq \emptyset$, i.e., $\bigcap_{n\in\mathbb{N}}$ i-cl(j-int($X\setminus U_{\alpha_n}$)) $\neq \emptyset$. Thus {i-cl(j-int($X\setminus U_\alpha$)) : $\alpha \in \Delta$} is a family of (i, j)-regular closed subsets of X satisfies $\bigcap_{n\in\mathbb{N}}$ i-cl(j-int($X\setminus U_{\alpha_n}$)) $\neq \emptyset$ for any countable subfamily {i-cl(j-int($X\setminus U_{\alpha_n}$)) : $n \in \mathbb{N}$}. By (iii), the intersection $\bigcap_{\alpha\in\Delta}$ i-cl(A_α) $\neq \emptyset$ for each (j, i)-regular open subset A_α of X with $A_\alpha \supseteq$ i-cl(j-int($X\setminus U_\alpha$)), and so $X\setminus\bigcup_{\alpha\in\Delta}$ i-int($X\setminus A_\alpha$) $\neq \emptyset$, i.e., $X \neq \bigcup_{\alpha\in\Delta}$ i-int($X\setminus A_\alpha$). Since $X\setminus A_\alpha \subseteq$ i-int(j-cl(U_α)), then $X\setminus A_\alpha$ is any (j, i)-regular closed subset of X with $X \neq \bigcup_{\alpha\in\Delta}$ i-int($X\setminus A_\alpha$). It is a contradiction with the fact that $\{U_\alpha : \alpha \in \Delta\}$ is an (i, j)-regular cover of X. Therefore X is (i, j)-nearly regular-Lindelöf.

(ii) ⇔ (iv): Let $\{C_\alpha : \alpha \in \Delta\}$ be a family of i-closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j, i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta}$ i-cl(A_α) $= \emptyset$. Then {i-cl(j-int(C_α)) : $\alpha \in \Delta$} is a family of (i, j)-regular closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j, i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha \supseteq$ i-cl(j-int(C_α)) and $\bigcap_{\alpha\in\Delta}$ i-cl(A_α) $= \emptyset$. By (ii), there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n\in\mathbb{N}}$ i-cl(j-int(C_{α_n})) $= \emptyset$.

Conversely, let $\{C_\alpha : \alpha \in \Delta\}$ be a family of (i, j)-regular closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j, i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta}$ i-cl(A_α) $= \emptyset$. Since $\{C_\alpha : \alpha \in \Delta\}$ is also a family of i-closed subsets of X, by (iv), there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n\in\mathbb{N}}$ i-cl(j-int(C_{α_n})) $= \emptyset$. Since for each α, C_α is (i, j)-regular closed subsets of X, then $\bigcap_{n\in\mathbb{N}} C_{\alpha_n} = \emptyset$.

(ii) ⇔ (iii) and (iv) ⇔ (v): Straightforwards. □

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