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Viscoelastic Discontinuity Analysis for Impact of Viscoelastic Slug and Elastic Rod (Standard Linear Solid Model)

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ABSTRACT

The study is about impact of a short viscoelastic slug on a stationary semi-infinite elastic rod. The viscoelastic material is modeled as standard linear solid which involve three material parameters and the motion is treated as one-dimensional. We first establish the governing equations pertaining to the impact of viscoelastic materials subject to certain boundary conditions for the case when a viscoelastic slug moving at a speed *V* impacts a semi-infinite stationary viscoelastic rod. The objective is to investigate how the viscosity time constants in the slug gives rise to different interface stresses and interface velocities following wave transmission in the slug. After modeling the impact, we solve the governing system of partial differential equations in the Laplace transform domain. We invert the Laplace transformed solution numerically to obtain the stresses and velocities. In inverting the Laplace transformed equations; we used the complex inversion formula (Bromwich contour). Then we introduce the moving viscoelastic discontinuity technique used by Achenbach, J.D. who considered in detail the propagation of the wavefront and dynamical and kinematical condition at the wavefront to predict the behaviour of the viscoelastic slug. Finally, we discussed the relationship between the viscosity time constants, ratios of acoustic impedances and the results of the viscoelastic impacts obtained numerically and the predictions acquired using viscoelastic discontinuity analysis.

Keyword: Ratio of acoustic impedances, Viscoelastic discontinuity, Viscosity time constants

INTRODUCTION

There are materials for which a suddenly applied and maintained state of uniform stress induces an instantaneous deformation followed by a flow process which may or may not be limited in magnitude as time grows [1]. These materials are said to exhibit both an instantaneous elasticity effect and a creep characteristic. This behavior clearly cannot be described by either elasticity or viscosity theories alone as it combines features of each and is called viscoelastic. Viscoelasticity is a generalization of elasticity and viscosity. The ideal linear elastic element is the spring whilst the ideal linear viscous element is the dashpot. Energy is stored in springs as elastic strain energy and energy is dissipated in a dashpot as heat [2].

In general there are four types of analysis for low speed collisions, associated with particle impact, rigid body impact, transverse impact on flexible bodies (i.e. transverse wave propagation or vibrations) and axial impact on flexible bodies (i.e. longitudinal wave propagation) [3]. We are interested in the latter impact where it generates longitudinal waves which affects the dynamic analysis of the bodies. The unifying characteristic of waves is propagation of disturbances through the medium. The properties of the medium that affect the waves and determine the speeds of propagation are density ρ and Young's modulus E, of deformability.

In this paper, we predict the stress and velocity at the interface after a moving slug impacts a stationary semi-infinite rod using viscoelatic discontinuity analysis. If the stress at the interface becomes tensile and the velocity changes its sign, then the slug and the rod part company. If the stress at the interface is compressive after the impact, then the slug and the rod remain in contact. In the elastic impact considered by R.P. Menday [4], the stress becomes tensile if the ratio of acoustic impedances $z < 1$ and the stress becomes compressive if the ratio of acoustic impedances $z < 1$ when the wave set up in the slug by the impact has returned to the slug/rod interface. In this viscoelastic impact we investigate how the viscosity time constants in the slug and in the rod give rise to different interface stresses and interface velocities following wave transmission in the slug.

MATHEMATICAL MODEL OF VISCOELASTIC IMPACT

We model the impact by having a finite length slug, moving with speed V, impacting a stationary semi-infinite rod and we solve the problem in the Laplace transform domain for the general case of viscoelastic slug and rod. In deriving the numerical solution, we firstly consider the slug is elastic and the rod is viscoelastic. We model the viscoelastic material as a standard linear solid (Figure 1). Secondly, we consider the slug is viscoelastic and the rod is elastic and lastly, we consider both materials are viscoelastic. We then numerically compute the interface stress and interface velocity using the complex inversion formula [5].

GOVERNING EQUATIONS AND GENERAL SOLUTIONS

Let $\breve{u}$ and $\breve{\breve{u}}$ be additional displacements in the slug and in the rod following the impact respectively, $\breve{\sigma}$ and be the stress in the slug and in the rod respectively, and be the young modulus in the slug and in the rod respectively, E and $\overline{E}$ be the density in the slug and in the rod respectively. The quantities $\breve{u}$, $\breve{\breve{u}}$, $\breve{\eta}$ and $\breve{\breve{\eta}}$ are material constants with dimension of time where the $\smile$ notation indicates dimensional variables. We choose the origin of coordinates at the center of the interface and axis along the axis $0\breve{X}$ of the rod and we assume the impact takes place at time $\breve{t} = 0$. When a slug moving at a speed V, impacts the rod at time $\breve{t} = 0$ and at $\breve{X} = 0$, we write the position at time $\breve{t}$ of the cross-section of the slug which was at location at time $\breve{t} = 0$ as

$$\breve{x}\left(\breve{X},\breve{t}\right) = \breve{X} + V\breve{t} + \breve{u}\left(\breve{X},\breve{t}\right) \quad \text{for} \quad -h_s \leq \breve{X} \leq 0 \tag{1}$$

And the cross-section of the rod which was at location $\breve{\breve{X}}$ at time $\breve{t} = 0$ as

$$\breve{\breve{x}}\left(\breve{\breve{X}},\breve{t}\right) = \breve{\breve{X}} + \breve{\breve{u}}\left(\breve{\breve{X}},\breve{t}\right) \quad \text{for} \quad \breve{\breve{X}} \geq 0 \tag{2}$$

Then the equation of motion in the slug is

$$\frac{\partial \breve{\sigma}}{\partial \breve{X}} = \rho \frac{\partial^2 \breve{u}}{\partial \breve{t}^2} \tag{3}$$

And the equation of motion in the rod is

$$\frac{\partial \breve{\bar{\sigma}}}{\partial \breve{\bar{X}}} = \bar{\rho} \frac{\partial^2 \breve{\bar{u}}}{\partial \breve{t}^2} \tag{4}$$

We model the slug and rod as a standard linear solid model so that the equation of viscoelastic stress related to $\breve{u}$ in the slug is

$$\breve{\sigma} + \breve{\mu}\frac{\partial \breve{\sigma}}{\partial \breve{t}} = E\left(\frac{\partial \breve{u}}{\partial \breve{X}} + \breve{\eta}\frac{\partial^2 \breve{u}}{\partial \breve{t}\partial \breve{X}}\right) \tag{5}$$

And the equation of viscoelastic stress related to $\breve{\bar{u}}$ in the rod is

$$\breve{\bar{\sigma}} + \breve{\bar{\mu}}\frac{\partial \breve{\bar{\sigma}}}{\partial \breve{t}} = \bar{E}\left(\frac{\partial \breve{\bar{u}}}{\partial \breve{\bar{X}}} + \breve{\bar{\eta}}\frac{\partial^2 \breve{\bar{u}}}{\partial \breve{t}\partial \breve{\bar{X}}}\right) \tag{6}$$

We now define the non-dimensional quantities $x, \bar{x}, X, t, u, \bar{u}, \sigma, \bar{\sigma}, \mu, \eta, \bar{\mu}, \bar{\eta}$ by the non-dimensionalising scheme below

$$\breve{X} = h_s X,\quad \breve{x} = h_s x,\quad \breve{t} = \frac{h_s}{c}t,\quad \breve{u} = \frac{V}{c}h_s u,\quad \breve{\bar{u}} = \frac{V}{\bar{c}}h_s \bar{u},\quad \breve{w} = \frac{V}{c}h_s w,\quad \breve{\sigma} = E\sigma$$

$$\breve{\bar{\sigma}} = E\bar{\sigma},\quad \breve{\bar{x}} = h_s \bar{x},\quad \breve{\bar{\eta}} = \frac{h_s}{\bar{c}}\bar{\eta},\quad \breve{\bar{\mu}} = \frac{h_s}{\bar{c}}\bar{\mu},\quad \breve{\eta} = \frac{h_s}{c}\breve{\eta},\quad \breve{\mu} = \frac{h_s}{c}\breve{\mu},\quad \breve{\bar{\eta}} > \breve{\bar{\mu}},\quad \breve{\eta} > \breve{\mu}, \tag{7}$$

Where

$$c^2 = \frac{E}{\rho},\quad \bar{c}^2 = \frac{\bar{E}}{\bar{\rho}},\quad z = \frac{\rho c}{\bar{\rho}\bar{c}} \text{ and } \alpha = \frac{c}{\bar{c}}.$$

If we now use (7) to non-dimensionalize equations (1) and (2) for $\breve{x}$ and $\breve{\bar{x}}$, the non-dimensional displacements $\breve{x}$ and $\breve{\bar{x}}$ are given by

$$x = X + \frac{V}{c}(t + u(X,t)) \tag{8}$$

$$\bar{x} = X + \frac{V}{c}\alpha \bar{u}(X,t) \tag{9}$$

We then non-dimensionalze (3) – (6) to obtain the non-dimensional equation of motion and stress-strain relations

$$\frac{\partial \sigma}{\partial X} = \frac{V}{c}\frac{\partial^2 u}{\partial t^2} \tag{10}$$

$$\frac{\partial \bar{\sigma}}{\partial X} = \alpha^2 \frac{V}{\bar{c}}\frac{\partial^2 u}{\partial t^2} \tag{11}$$

$$\sigma + \mu\frac{\partial\sigma}{\partial t} = \frac{V}{c}\left(\frac{\partial u}{\partial X} + \eta\frac{\partial^2 u}{\partial t\partial X}\right) \tag{12}$$

$$\bar{\sigma} + \alpha\bar{\mu}\frac{\partial\bar{\sigma}}{\partial t} = \frac{V}{\bar{c}}\left(\frac{\partial\bar{u}}{\partial X} + \alpha\bar{\eta}\frac{\partial^2\bar{u}}{\partial t\partial X}\right) \tag{13}$$

In order to solve for the additional displacements u and $\bar{u}$ of the waves propagating in the slug and the rod, we take Laplace transforms of the equations (10) – (13) with respect to and solve the differential equations for the transformed displacement $\hat{u}$ and $\hat{\bar{u}}$ in the s domain. Taking the Laplace transform of the non-dimensionalized equations (10) and (11), after differentiating (12) with respect to X, gives

$$\frac{\partial\hat{\sigma}}{\partial X} = \frac{V}{c}\hat{u}s^2 \tag{14}$$

$$\frac{\partial\hat{\sigma}}{\partial X} = \frac{V}{c}\frac{\partial^2\hat{u}}{\partial X^2}\beta^2(s) \tag{15}$$

Where

$$\beta^2(s) = \frac{1+\eta s}{1+\mu s} \quad \text{and} \quad \bar{\beta}^2(s) = \frac{1+\alpha\bar{\eta}s}{1+\alpha\bar{\mu}s}$$

Equating (14) and (15) yields the differential equation below for $\hat{u}(x,s)$

$$\beta^2(s)\frac{\partial^2\hat{u}}{\partial X^2} - \hat{u}s^2 = 0 \tag{16}$$

Solving (16), we obtain the general solution for the transform of the additional displacement $\hat{u}(x,s)$ in the slug,

$$\hat{u}(X,s) = a(s)e^{\frac{sX}{\beta(s)}} + b(s)e^{-\frac{sX}{\beta(s)}} \tag{17}$$

Repeating the same process for the equations (11) and (13) gives the general solution $\hat{\bar{u}}(x,s)$ for the transform of the additional displacement in the rod,

$$\hat{\bar{u}}(X,s) = d(s)e^{\frac{\alpha sX}{\bar{\beta}(s)}} + f(s)e^{-\frac{\alpha sX}{\bar{\beta}(s)}} \tag{18}$$

BOUNDARY CONDITIONS

In order to find *a(s), b(s), d(s)* and *f(s)* in (17) and (18), we apply the boundary conditions described below.

1. The interface conditions state that the particle velocity in the slug and in the rod has to be the same at $\breve{X} = 0$ that is

$$V + \frac{\partial\breve{u}}{\partial\breve{t}}(0,\breve{t}) = \frac{\partial\breve{\bar{u}}}{\partial\breve{t}}(0,\breve{t}) \tag{19}$$

In non-dimensional form, the above equation becomes

$$1 + \frac{\partial u}{\partial t} = \alpha\frac{\partial\bar{u}}{\partial t} \tag{20}$$

Then we Laplace transform equation (20) and substitute from (17) and (18), we obtain

$$\frac{1}{s} + s[a(s) + b(s)] = \alpha s[d(s) + f(s)] \tag{21}$$

2. At the interface $\check{X} = 0$, the stress in the slug and in the rod must be the same so we have

$$\check{\sigma} = \check{\bar{\sigma}} \tag{22}$$

If we non-dimensionalize and take Laplace transform of the above equation we have

$$E\left(\frac{V}{c}\frac{\partial \hat{u}}{\partial X}\beta^2(s)\right) = \bar{E}\left(\frac{V}{\bar{c}}\frac{\partial \hat{\bar{u}}}{\partial X}\bar{\beta}^2(s)\right) \tag{23}$$

Since

$$\hat{\sigma} = \frac{V}{c}\frac{\partial \hat{u}}{\partial X}\beta^2(s) \ \text{ and } \ \hat{\bar{\sigma}} = \frac{V}{\bar{c}}\frac{\partial \hat{\bar{u}}}{\partial X}\bar{\beta}^2(s) \tag{24}$$

Substituting the derivatives of (17) and (18) at $\check{X} = 0$ into the equation (23), gives

$$z\beta(s)[a(s) - b(s)] = \alpha\bar{\beta}(s)[d(s) - f(s)] \tag{25}$$

3. When the wave in the slug reaches the boundary X=-1, the stress in the slug at $\check{X} = -h_s$ is zero that is $\check{\sigma} = 0$

   Non-dimensionalize and take Laplace transform of the above equation to obtain

$$E\left(\frac{V}{c}\frac{\partial \hat{u}}{\partial X}\beta^2(s)\right) = 0 \tag{26}$$

Substituting (17) at X = -1 into the above equation, we obtain

$$s\rho cV\beta(s)\left(a(s)e^{-\frac{s}{\beta(s)}} - b(s)e^{\frac{s}{\beta(s)}}\right) = 0 \tag{27}$$

4. The stress in the rod at $X = ah_s$ is zero, that is $\check{\bar{\sigma}} = 0$

   Non-dimensionalize and take Laplace transform of the above equation to obtain

$$\bar{E}\left(\frac{V}{\bar{c}}\frac{\partial \hat{\bar{u}}}{\partial X}\bar{\beta}^2(s)\right) = 0$$

Substituting (18) at $X = a$ into the above equation, we obtain

$$s\alpha\bar{\rho}\bar{c}V\bar{\beta}(s)\left(d(s)e^{\frac{\alpha as}{\bar{\beta}(s)}} - f(s)e^{-\frac{\alpha as}{\bar{\beta}(s)}}\right) = 0 \tag{28}$$

Equations (21), (26), (27) and (28) give four equations in the four unknowns *a(s), d(s)*, and f*(s)*. Solving for these unknowns and substituting into (17) and (18) give the additional displacements $\hat{u}$ and $\hat{\bar{u}}$ in the Laplace transform domain,

$$\hat{u} = -\frac{\sinh\left(\frac{\alpha as}{\bar{\beta}(s)}\right)\cosh\left(\frac{s}{\beta(s)}(1+X)\right)}{s^2\left(z\frac{\beta(s)}{\bar{\beta}(s)}\sinh\left(\frac{s}{\beta(s)}\right)\cosh\left(\frac{\alpha as}{\bar{\beta}(s)}\right) + \sinh\left(\frac{\alpha as}{\bar{\beta}(s)}\right)\cosh\left(\frac{s}{\beta(s)}\right)\right)} \tag{29}$$

$$\hat{u} = \frac{z\beta(s)\sinh\left(\frac{s}{\beta(s)}\right)\cosh\left(\frac{\alpha s}{\bar{\beta}(s)}(X-a)\right)}{s^2\left(z\beta(s)\sinh\left(\frac{s}{\beta(s)}\right)\cosh\left(\frac{\alpha as}{\bar{\beta}(s)}\right)+\bar{\beta}(s)\sinh\left(\frac{\alpha as}{\bar{\beta}(s)}\right)\cosh\left(\frac{s}{\beta(s)}\right)\right)} \tag{30}$$

Having found the general solution (29) and (30) for the displacements in the slug and the rod in the s-domain, we then can derive the equations for stress and the velocity in the slug and the rod. To do this we use the complex inversion formula to invert the transforms. As the general solution in the Laplace transform domain is particularly complicated, we consider its inversion in certain special cases. In solving for stress and velocity, we are considering a slug traveling at speed V which impacts semi-infinite rod. These solutions apply provided the interface stress remains compressive. If the stress at the interface becomes tensile, then the solution no longer valid since there can be no longer tensile stress at the interface. It is then necessary to modify the solution by introducing waves traveling away from the interface stress at zero. If the stress is compressive, then the slug and the rod remain in contact until such time as the stress drops to zero.

Firstly, we consider the general case when both the slug and the rod are viscoelastic. Then we apply the Bromwich contour to lay-out the calculation of the complex integrals along the contour and determine the poles and branch points. Secondly, we compute the residue of the simple pole and numerically compute the rest of the residues and the complex integrals. We consider the slug is viscoelastic and rod is elastic where $\eta \neq 0,\ \mu \neq 0,\ \bar{\eta} = 0,\ \bar{\mu} = 0$. Considering the case where the slug is viscoelastic and the rod is elastic, the Laplace transform of the stress in the slug is given by equation (24) and putting $\frac{v}{c} = 1$ and considering the general solution of the slug displacement equation (29), the solution (24) in the case when $a \to \infty$ gives

$$\hat{\sigma}(X,s) = -\frac{\beta(s)\sinh\left(\frac{s}{\beta(s)}(1+X)\right)}{s\left(z\frac{\beta(s)}{\bar{\beta}(s)}\sinh\left(\frac{s}{\beta(s)}\right)+\cosh\left(\frac{s}{\beta(s)}\right)\right)} \tag{31}$$

In order to find the stress as a function of time, we have to invert the solution (31) and we employ the complex inversion formula[4].

VISCOELASTIC DISCONTINUITY

Assume there are discontinuities in $\check{v}, \check{\varepsilon}$ *and* $\check{\sigma}$ and across the surface $\frac{dx}{dt} = U$. Let $v^-\varepsilon^-$ and σ^- denote velocity, strain and stress, respectively behind the moving surface U while $v^+\varepsilon^+$and σ^+ denote velocity, strain and stress, respectively ahead of the moving surface. As the surface moves, we consider the change of momentum between the times t and $t + \delta t$ where the velocity of the mass $\rho AU\delta t$ changes from v^+ to v^- to give the change as $\rho A\check{U}(\check{c}^- - \check{v}^+)\delta t$. The change of the momentum, must equal the impulse of the net force which gives

$$\left(\check{\sigma}^+ - \check{\sigma}^-\right)A\delta\check{t} = \rho A\check{U}\left(\check{v}^- - \check{v}^+\right)\delta\check{t} \tag{32}$$

We non-dimensionalise equation (32) and obtain

$$\left(\sigma^+ - \sigma^-\right) = \frac{V}{c}U\left(v^- - v^+\right) \tag{33}$$

$$-\frac{[\sigma]}{U} = \frac{V}{c}[v] \tag{34}$$

Considering equation (3.11) in Musa [6] and replacing $\overline{U}$ by U, we obtain

$$U\left[\frac{df}{dx}\right] + \left[\frac{df}{dt}\right] = 0 \tag{35}$$

Substituting f by u in equation (35) gives

$$-\frac{[v]}{U} = [\varepsilon] \tag{36}$$

Considering equation (3.16) in Musa [6] for the non-dimensional stress-strain relationof viscoelastic material and replacing $\overline{\eta}, \overline{\mu}$, and $\overline{c}$ by η, μ and c respectively, we obtain

$$\mu[\sigma] = \frac{V}{c}\eta[\varepsilon] \tag{37}$$

Eliminating $[\sigma]$ and $[\varepsilon]$ in equations (34), (36) and (37) gives

$$\left(\mu U - \frac{\eta}{U}\right)[v] = 0 \tag{38}$$

Which gives $U = \pm\sqrt{\frac{\eta}{\mu}}$. Considering from equation (3.19) to (3.23) in Musa [6], we obtain

$$-U[v] + \mu U^2\left[\frac{\partial v}{\partial X}\right] - 2\mu U\frac{\delta}{\delta t}[v] = -\frac{[v]}{U} + \eta\left[\frac{\partial v}{\partial X}\right] \tag{39}$$

Further simplification, reduces equation (39) to

$$\frac{\delta}{\delta t}[v] = \frac{1}{2\mu}\left(\frac{1}{U^2} - 1\right)[v] = \frac{1}{2}\left(\frac{1}{\eta} - \frac{1}{\mu}\right)[v] \tag{40}$$

Equation (40) integrates to give the variations of jump in $[v]$ as we move with the front, in the form

$$[v] = [v]_0 e^{\frac{\mu-\eta}{2\eta\mu}t} \tag{41}$$

Where $[v]_0$ is the value of the jump at time $t = 0$. It follows from equations (34) and (36) that

$$[\varepsilon] = [\varepsilon]_0 e^{\frac{\mu-\eta}{2\eta\mu}t} \quad \text{and} \quad [\sigma] = [\sigma]_0 e^{\frac{\mu-\eta}{2\eta\mu}t} \tag{42}$$

Which agrees with result of Morrison [7].

Applying the result to the impact problem where we solve the viscoelastic equations in the rod and slug subject to the boundary conditions

$$1 + v = \alpha\bar{v} \quad \text{and} \quad \alpha z[\sigma]_0 = [\bar{\sigma}]_0 \tag{43}$$

At impact, we will have a discontinuity in velocity in both slug and rod and a discontinuity in stress given by (43)

$$1 + [v]_0 = \alpha[\bar{v}]_0 \quad \text{and} \quad \alpha z[\sigma]_0 = [\bar{\sigma}]_0 \tag{44}$$

There will be a discontinuity in the slug along $\frac{dx}{dt} = -U$ and in the rod $\frac{d\bar{x}}{dt} = \bar{U}$ and from the results in equation (33), we obtain

$$[\sigma]_0 = \frac{UV}{c}[v]_0 \tag{45}$$

$$[\bar{\sigma}]_0 = -\alpha^2 \frac{\bar{U}V}{\bar{c}}[\bar{v}]_0 \tag{46}$$

Where $U = \sqrt{\frac{\eta}{\mu}}$ and $\bar{U} = \sqrt{\frac{\bar{\eta}}{\bar{\mu}}}$. Eliminating $[\bar{v}]_0$ gives

$$[v]_0 = -\frac{1}{1+z^*} \tag{47}$$

$$[\sigma]_0 = -\frac{V}{c}\frac{\sqrt{\frac{\eta}{\mu}}}{(1+z^*)} \tag{48}$$

and in the rod

$$[\bar{v}]_0 = \frac{z^*}{\alpha(1+z^*)} \tag{49}$$

$$[\bar{\sigma}]_0 = -\frac{V}{\bar{c}}\frac{z\sqrt{\frac{\eta}{\mu}}}{(1+z^*)} \tag{50}$$

Where $z^* = z\sqrt{\frac{\eta\bar{\mu}}{\bar{\eta}\mu}}$. These results agree with results obtained by Musa[6] using the limit theorem(initial value theorem) for the Laplace Transform solutions. As the discontinuity moves in the slug and reaches the free end(X=-1) $t = \frac{1}{\sqrt{\frac{\eta}{\mu}}} = \sqrt{\frac{\mu}{\eta}}$ at time at which time the magnitudes will be

$$[v] = -\frac{1}{1+z^*}e^{-\frac{\eta-\mu}{2\eta\sqrt{\eta\mu}}} \tag{51}$$

$$[\sigma] = -\frac{V}{c}\frac{\sqrt{\frac{\eta}{\mu}}}{1+z^*}e^{-\frac{\eta-\mu}{2\eta\sqrt{\eta\mu}}} \tag{52}$$

The stress-free condition at X=-1 requires a reflected pulse to travel back along $\frac{dx}{dt} = U$ with its stress discontinuity being equal and opposite to that given by (52). Since the discontinuity in v is now related to that in σ by equation (33), is still given by equation (51). This reflected pulse will reach the interface at time $t = 2\sqrt{\frac{\mu}{\eta}}$ at which time the jumps in v and σ are given by

$$[v] = -\frac{1}{1+z^*}e^{-\frac{\eta-\mu}{\eta\sqrt{\eta\mu}}} \tag{53}$$

$$[\sigma] = -\frac{V}{c}\frac{\sqrt{\frac{\eta}{\mu}}}{1+z^*}e^{-\frac{\eta-\mu}{\eta\sqrt{\eta\mu}}} \tag{54}$$

IMPACT OF VISCOELASTIC SLUG AND ELASTIC ROD

In order to make the rod elastic, we let $\bar{\beta} = 1$ or $\bar{\eta} = 0$ and $.\bar{\mu} = 0$ Then the general solution in equation (29)

$$= -\frac{\cosh\left(\frac{s}{\beta(s)}(1+X)\right)}{s^2\left(z\beta(s)\sinh\left(\frac{s}{\beta(s)}\right) + \cosh\left(\frac{s}{\beta(s)}\right)\right)} \tag{55}$$

Table 1

z	P_{LTIS}	P_{LTIV}	$\frac{\eta}{\mu}$	z^*	P_{ISD}	P_{IPV}
0.33	-0.75	0.25	2	0.47	-0.961	0.2
0.67	-0.6	0.4	2	0.94	-0.728	0.49
0.9	-0.526	0.47	2	1.27	-0.622	0.56
1	-0.5	0.5	2	1.41	-0.586	0.59
1.2	-0.454	0.55	2	1.7	-0.524	0.63
0.33	-0.75	0.25	5	0.75	-1.28	0.43
0.67	-0.6	0.4	5	1.49	-0.897	0.6
0.9	-0.526	0.47	5	2.01	-0.742	0.67
1	-0.5	0.5	5	2.24	-0.691	0.69
1.2	-0.454	0.55	5	2.68	-0.607	0.73

P_{ISD}= Predicted initial stress discontinuity and P_{IPV}=Predicted initial particle velocity at the interface at $t=0$ in the viscoelastic slug. P_{LTS} and P_{LTIV} are predicted initial stress discontinuity and initial velocity respectively, based on the long time ratios of acoustic impedances z

The Laplace transform of the stressing the viscoelastic slug is given by

$$\hat{\sigma}(X,s) = -\frac{\beta(s)\sinh\left(\frac{s}{\beta(s)}(1+X)\right)}{s\left(z\beta(s)\sinh\left(\frac{s}{\beta(s)}\right) + \cosh\left(\frac{s}{\beta(s)}\right)\right)} \tag{56}$$

In this case where the slug is viscoelastic and the rod is elastic, the effective ratio of acoustic impedances is $z^* = z\sqrt{\frac{\eta}{\mu}}$. The predicted initial interface stress is

$$[\sigma]_0 = -\frac{\sqrt{\frac{\eta}{\mu}}}{\left(1+z^*\right)} \tag{57}$$

The predicted initial interface stress discontinuities are shown in Table 1. We also can predict the stress discontinuities after the wave rebounds at X = -1 and reaches the interface using Appendix I in Musa [6]. By replacing $z^* = z\sqrt{\frac{\eta}{\mu}}$ and we obtain

$$[\sigma]_1 = -\left(\frac{2}{z^* + 1}\right)[\sigma]_0 e^{-\frac{\eta - \mu}{\eta\sqrt{\eta\mu}}} \tag{58}$$

As the interface stress discontinuity at non-dimensional time $t = 2\sqrt{\frac{\mu}{\eta}}$ where $[\sigma]_0 e^{-\frac{\eta-\mu}{\eta\sqrt{\eta\mu}}}$ is the incoming wave. The calculated predicted interface stress discontinuities after the wave rebounds at X = -1 are shown in Table 2. We also can predict the initial interface velocity using equation (47) and obtain

$$1 + [v]_0 = 1 - \frac{1}{1 + z^*} = \frac{z^*}{1 + z^*} \tag{59}$$

and the velocity discontinuities after the wave rebounds at X = -1 and reaches the interface using Appendix I in Musa [6]. By replacing $z^* = z\sqrt{\frac{\eta}{\mu}}$ and we obtain

$$[v]_1 = -\left(\frac{2z^*}{z^* + 1}\right)[v]_0 e^{-\frac{\eta - \mu}{\eta\sqrt{\eta\mu}}} \tag{60}$$

As the interface velocity jump at non-dimensional time $t = 2\sqrt{\frac{\mu}{\eta}}$ and X = 0 where $[v]_0 e^{-\frac{\eta-\mu}{\eta\sqrt{\eta\mu}}}$ is the incoming wave.

The calculated initial interface velocity and the interface velocity after the pulse rebounds at X = -1 are shown in Table 2.

Table 2

z	$\frac{\eta}{\mu}$	μ	z^*	P_{SIJ}	P_{V1J}		$\frac{\eta}{\mu}$	μ	z^*	P_{SIJ}	P_{V1J}
0.333	2	10	0.471	1.261	-0.42		5	10	0.745	1.411	-0.472
0.667	2	10	0.943	0.723	-0.483		5	10	1.49	0.695	-0.463
0.9	2	10	1.273	0.529	-0.476		5	10	2.012	0.475	-0.428
1	2	10	1.414	0.469	-0.468		5	10	2.236	0.412	-0.412
1.2	2	10	1.697	0.375	-0.451		5	10	2.683	0.318	-0.383
0.333	2	1	0.471	0.917	-0.306		5	1	0.745	1.026	-0.342
0.667	2	1	0.943	0.526	-0.351		5	1	1.49	0.504	-0.336
0.9	2	1	1.273	0.384	-0.346		5	1	2.012	0.344	-0.31
1	2	1	1.414	0.341	-0.341		5	1	2.236	0.299	-0.299
1.2	2	1	1.697	0.273	-0.328		5	1	2.683	0.23	-0.277

P_{SIJ}= Predicted stress jump and P_{VIJ}= Predicted velocity jump at the interface after the wave first rebound X = -1 at in the slug for several ratios of effective acoustic impedances Z^*

Results from numerically calculated interface stress and velelocity for several ratios of acoustic impedances and viscosity time constants [7] are shown in Figure 2, 3, 4 and 5.

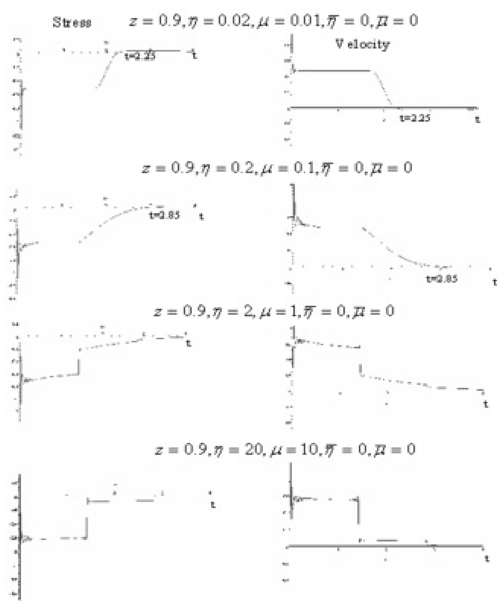

Figure 2

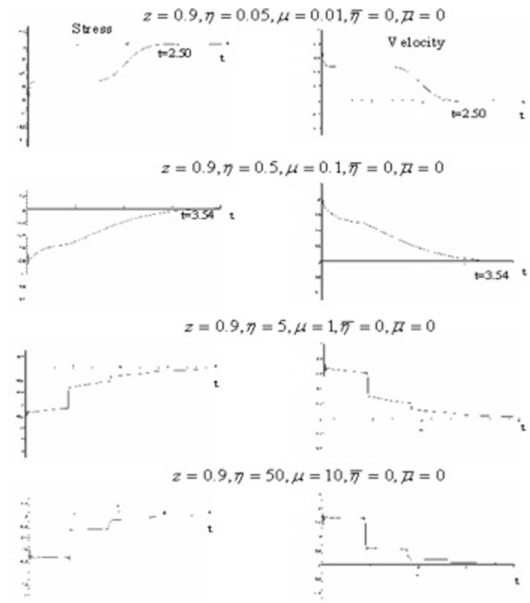

Figure 3

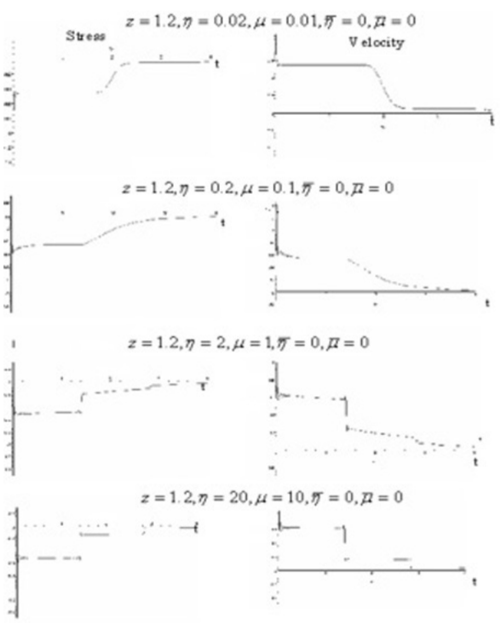

Figure 4

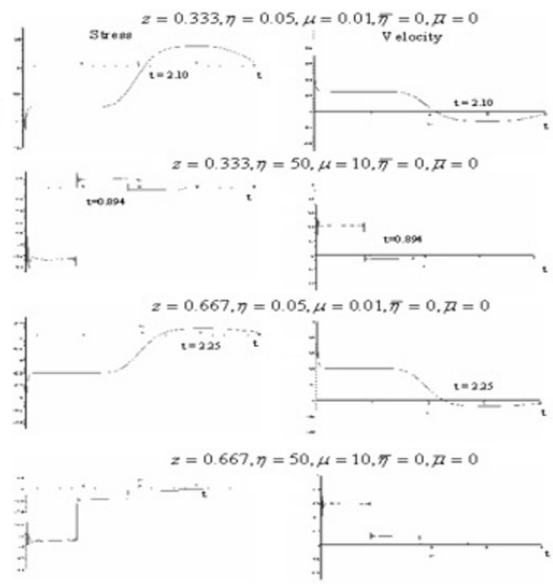

Figure 5

DISCUSSION AND CONCLUSIONS

In the elastic impact studied by R.P. Menday[4], the interface stress at time $t = 2$ is tensile for $z < 1$ and compressive for $z < 1$ (where $t = 2$ is the time taken for the impact wave to travel backward and forward in the slug). However in this impact, the viscosity time constants of the materials and the ratios of acoustic impedances z play significant role in determining the stress. As the viscosity time constants in the slug increases, it takes longer time for the stress of the slug to become tensile at the interface as depicted in Figure 2 and Figure 3.

When the ratios of acoustic impedances z increase, the time taken for the stress to become tensile also increases as shown in Figure 2 – Figure 5. The time taken for the stress to become tensile for increasing ratio of acoustic impedances z is also found in R.P. Menday[4]. However in this impact, we find that there are two aspects which prolong the time for the stress to become tensile. The first aspect is the ratios of acoustic impedances z and the second aspect is the viscosity time constants in the slug.

In Figure 3 shows that there is an increase in stress and a big drop in velocity for $0 \leq t \leq 0.1$ when the viscosity time constants are $\eta = 0.05,\ \mu = 0.01$, the increase is from -0.76 to and the drop is from 0.68 to 0.48, respectively. Moreover when $\eta = 0.5,\ \mu = 0.1$, the increase is from -0.84 to -0.68 in stress and the drop is from 0.76 to in 0.68 velocity. However there is a small increase in stress and a small decrease in velocity when viscosity time constants $\eta = 5,\ \mu = 1$ and almost no increase nor decrease in stress and velocity respectively when $\eta = 50,\ \mu = 10$. This is because there are two different values of effective ratios of acoustic impedances. The long time acoustic impedance is $z = 0.9$ when the viscosity time constant are small whereas for the short effective ratio of acoustic impedances is $z^* = 2.012$ when the viscosity time constants are large. The same trend of results are also shown in Figures 2, 4 and 5. Figure 2 shows that almost all the initial interface stress and initial interface velocity curves fluctuate as t approaches zero. As t increases, the curves settle down and show that the initial interface stress discontinuity between the long term initial stress, -0.526 and short term initial interface stress -0.622 for $z = 0.9$ as predicted in Table 1. In all the results show that the actual initial interface stress and interface velocity are within the predicted initial interface stress and velocity discontinuities based on the long time acoustic impedance z and the short time acoustic impedance z^* in Table 1. As the viscosity time constants increase the predicted value approximates the initial interface stress and initial interface velocity better. This results can be explained by the creep and the stress relaxation tests in Appendix A[6]. In the stress relaxation test, the stress response decreasing rapidly when the viscosity time constants are $\eta = 0.05, \mu = 0.01$ and the viscoelastic material behaves like the long time elastic material whereas when the viscosity time constants are $\eta = 50, \mu = 10$, the interface stress remain constant and the viscoelastic material behaves like the short time elastic material[6].

Table 2 shows that the interface velocity jump at non-dimensional time $t = 2\sqrt{\frac{\mu}{\eta}}$ for z = 0.9 is -0.476 for viscosity time constants $\eta = 20, \mu = 10$ and is -0.346 for viscosity time constant $\eta = 2$, and $\mu = 1$. The actual interface velocity jump in Figure 2 for z =0.9 is about -0.46 for viscosity time constants $\eta = 20, \mu = 10$ and -0.34 when the viscosity time constants $\eta = 2, \mu = 1$. The predicted interface velocity at time $t = 2\sqrt{\frac{\mu}{\eta}}$ for z = 0.333 when $\eta = 20, \mu = 10$ is -0.472 and the actual interface velocity is about -0.48.

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