

Entropy Weight with Interval Type-2 Fuzzy Sets (Pemberat Entropi dengan Selang Set Kabur Jenis-2)

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ABSTRACT

The design of weight is one of the important parts in fuzzy decision making, as it would have a deep effect on the evaluation results. Entropy is one of the weight measures based on objective evaluation. Non-probabilistic-type entropy measures for fuzzy set and interval type-2 fuzzy sets (IT2FS) have been developed and applied to weight measure. Since the entropy for IT2FS for decision making yet to be explored, hence this paper proposes a new objective weight method by using entropy weight method for multiple attribute decision making (MADM). This paper utilizes the nature of IT2FS concept in the evaluation process to assess the attribute weight based on the credibility of data. An example is presented to demonstrate the feasibility of the new method in decision making. The entropy measure of interval type-2 fuzzy sets yield flexible judgment and could be applied in decision making environment.

Keywords: Objective weight, entropy weight, multiple attribute decision making, type-2 fuzzy sets, interval type-2 fuzzy sets.

ABSTRAK

Reka bentuk pemberat adalah salah satu bahagian penting didalam membuat keputusan kabur kerana ia akan memberi kesan yang mendalam kepada keputusan penilaian. Entropi adalah salah satu ukuran pemberat yang berdasarkan penilaian secara objektif. Sukatan entropi jenis tidak kebarangkalian untuk set kabur dan selang set kabur jenis-2 (IT2FS) telah dibangunkan dan digunakan untuk sukatan pemberat. Memandangkan entropi untuk IT2FS bagi pembuat keputusan belum diterokai, oleh yang demikian kajian ini mencadangkan kaedah baru iaitu pemberat objektif baru dengan menggunakan kaedah pemberat entropi dengan pembuat keputusan multi-artibut (MADM). Kajian ini menggunakan ciri- ciri konsep IT2FS dalam proses penilaian untuk menilai berat atribut berdasarkan kredibiliti data. Satu contoh dikemukakan untuk menunjukkan kesesuaian kaedah baru dalam membuat keputusan. Ukuran selang entropi set kabur jenis-2 menghasilkan pertimbangan yang fleksibel dan boleh digunakan dalam membuat keputusan persekitaran.

Katakunci: Pemberat objektif, pemberat entropi, pembuat keputusan multi-atribut (MADM), set kabur jenis-2, selang set kabur jenis-2

INTRODUCTION

Weight in multiple attribute decision making (MADM) can be divided into two groups which are subjective weight and objective weight. Subjective weight can reflect the subjective judgment or intuition of the decision makers (DM), and they can be obtained based on preference information of the attributes given by the DM through interviews, questionnaires or trade-off interrogation directly (Chen & Lee, 2011). Objective weight can be obtained from the objective information such as decision matrix through mathematics models (Hwang and Yoon, 1981).

The most popular method to obtain objective weights is entropy method (Hwang and Yoon, 1981). The conception of 'entropy' firstly appeared in thermodynamics, and was used to describe the process of a campaign irreversible phenomenon. Later 'entropy' was introduced to the information theory by Shannon to measure the disorder degree of system (Shannon, 1948). The entropy value as a measure of variability and randomness, high information entropy indicates high variability, which means high levels of uncertainty in a system (Wang *et al.*, 2009). Then, in 1965, the entropy of a fuzzy set describes the fuzziness degree of a fuzzy set and was first mentioned by Zadeh. Since then, many fuzzy entropy measures have been proposed. In 1972, De Luca and Termini introduced some axioms, which captured people's intuitive comprehension to describe the fuzziness degree of a fuzzy set. Then there have been various fuzzy entropy measures (Kosko, 1986; Yager, 1979; Pal & Pal, 1992), especially, Burillo and Bustince (1996) introduced the concept of entropy of Antanassove's intuitionistic fuzzy set and the interval-valued fuzzy set in 1996. Then in 2001, Szmiedt and Kacprzyk proposed a different concept for assessing the intuitionistic fuzzy (IF) entropy.

Since fuzzy sets (FS) or usually known as type-1 fuzzy sets (T1FS) has found not be able to handle all kinds of uncertainties appearing in real life problem domain (Saikit & Jaya, 2009). Therefore in 1975, Zadeh has introduced a type-2 fuzzy sets (T2FS) concepts that are more realible to handle all the uncertainties. However, T2 FS is difficult to understand and explain. Hence, Mendel and Liang (2000) introduced new concepts that are more easier to calculate. These new concepts are allowing the characterization of a type-2 fuzzy set with a superior membership function and an inferior membership function; these two functions can be represented each one by a type-1 fuzzy set membership function. The interval between these two functions represent the footprint of uncertainty (FOU), which is used to characterize a type-2 fuzzy set. Then in 2006, Mendel, John and Liu are introduced new concepts of interval type-2 fuzzy sets (IT2FS). The simplest T2FS are interval type-2 fuzzy sets whose elements' degree of membership are intervals with secondary membership degree of 1.0. If we can use interval type-2 fuzzy sets (Mendel, John, & Liu, 2006) for handling entropy weights problems, then there is room for more flexibility due to the fact that T2FS provide more flexibility to represent uncertainties than traditional T1FS (Zadeh, 1965).

To deal with the uncertainties problems, this paper presents a new method for entropy weight based on the concepts of IT2FS. This concept is totally different with the traditional entropy method. This proposed method provides a useful way to handle entropy weight for fuzzy multiple attributes group decision-making problems in a more flexible and more intelligent manner due to the fact that it uses interval type-2 fuzzy sets rather than traditional type-1 fuzzy sets to represent the evaluating values and the weights of attributes. The rest of this paper is organized as follows: in Section 2, we briefly review basic concepts of type-2 fuzzy sets and interval type-2 fuzzy sets and related theory behind it. In Section 3, we briefly review the concept of entropy weight. In Section 4, we introduce the new concept of entropy weight and applied it to weight measure. In section 5, a numerical example is performed into the new concepts of entropy weight and this paper concludes at the final section.

TYPE-2 FUZZY SETS AND INTERVAL TYPE-2 FUZZY SETS

This section is briefly review some definitions of type-2 fuzzy sets and interval type-2 fuzzy sets from Mendel *et al.* (2006).

Definition 1 (Mendel *et al.*, 2006)

A type-2 fuzzy set \tilde{A} in the universe of discourse X can be represented by a type-2 membership function $\mu_{\tilde{A}}$, shown as follows (Mendel *et al.*, 2006):

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1], 0 \leq \mu_{\tilde{A}}(x, u) \leq 1\}, \tag{1}$$

where J_x denotes an interval in $[0, 1]$. Moreover, the type-2 fuzzy set \tilde{A} also can be represented as follows:

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u), \tag{2}$$

where $J_x \subseteq [0, 1]$ and $\int \int$ denotes the union over all admissible x and u .

Definition 2 (Mendel *et al.*, 2006)

Let \tilde{A} be a type-2 fuzzy set in the universe of discourse X represented by the type-2 membership function $\mu_{\tilde{A}}$. If all $\mu_{\tilde{A}} = 1$ then \tilde{A} is called an interval type-2 fuzzy sets. An interval type-2 fuzzy set \tilde{A} can be regarded as a special case of a type-2 fuzzy set, represented as follows:

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} 1 / (x, u), \tag{3}$$

where $J_x \subseteq [0, 1]$

ENTROPY WEIGHT

Entropy weight is a parameter that describes how much different alternatives approach one another in respect to a certain attribute (Liu & Kong, 2005). Conversely, low information entropy is a sign of a highly organized system. In information theory, the entropy value can be calculated as Equation 4:

$$H(p_1, p_2, \dots, p_n) = - \sum_{j=1}^n p_j \ln p_j \tag{4}$$

where, H is the level of entropy, p_j is the probability of occurrence of event.

Szmidt and Kacprzyk (2001) proposed a new entropy method for IFS. In their paper, they proposed the IF entropy as a ratio of distances between the (F, F_{near}) and (F, F_{far}) . The expression given as follows:

$$E_{SK}(F) = \frac{(F, F_{near})}{(F, F_{far})} \tag{5}$$

where (F, F_{near}) is the distance from F to the nearer point F_{near} among positive ideal point and negative ideal point, and (F, F_{far}) is the distance from F to the farther point F_{far} among positive ideal point and negative ideal point. De Luca and Termini (1972) have already proposed the axioms of entropy for FSs. Szmidt and Kacprzyk (2001) then expressed IF entropy in the following definition:

Because the entropy concept of Szmidt and Kacprzyk constructs on distance, then the relative concepts of measuring distance for IFSs is shown. Szmidt and Kacprzyk (2000) introduced two different distance measures for IFS as follows: Let $X = \{x_1, x_2, \dots, x_n\}$. The Hamming distance between IFS A, B belonging to $IFSs(X)$ is defined by

$$d_{SK}^1(A, B) = \sum_{i=1}^n (|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|) \quad (6)$$

Clearly, $0 \leq d_{SK}^1(A, B) \leq n$. Then, they also defined another Euclidean distance between IFS A, B belonging to $IFSs(X)$ as follow:

$$d_{SK}^2(A, B) = \sqrt{\sum_{i=1}^n ((\mu_A(x_i) - \mu_B(x_i))^2 + (v_A(x_i) - v_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2)} \quad (7)$$

Clearly, $0 \leq d_{SK}^2(A, B) \leq n$.

Based on (4), the IF entropy measure that used Hamming distance and Euclidean distance given as below.

Definition 4 (Szmidt and Kacprzyk, 2001) Entropy for IFS A with n elements given as

IF entropy measure according to Hamming distance

$$(A1) E_{IFS}(A) = \frac{1}{n} \sum_{i=1}^n \left(\frac{|\mu_{near}(x_i) - \mu(x_i)| + |v_{near}(x_i) - v(x_i)| + |\pi_{near}(x_i) - \pi(x_i)|}{|\mu_{far}(x_i) - \mu(x_i)| + |v_{far}(x_i) - v(x_i)| + |\pi_{far}(x_i) - \pi(x_i)|} \right) \quad (8)$$

IF entropy measure according to Euclidean distance

$$(A2) E_{IFS}(A) = \frac{1}{n} \sum_{i=1}^n \left(\frac{\sqrt{(\mu_{near}(x_i) - \mu(x_i))^2 + (v_{near}(x_i) - v(x_i))^2 + (\pi_{near}(x_i) - \pi(x_i))^2}}{\sqrt{(\mu_{far}(x_i) - \mu(x_i))^2 + (v_{far}(x_i) - v(x_i))^2 + (\pi_{far}(x_i) - \pi(x_i))^2}} \right) \quad (9)$$

At this stage, we know that Szmidt and Kacprzyk (2000) made use of all three parameter to defined IF distance measures. Then a ratio-based measure of entropy was proposed based on IF distance measures (Szmidt and Kacprzyk, 2001).

STEPS OF OBJECTIVE WEIGHT METHOD BASED ON INTERVAL TYPE-2 FUZZY SET

In this paper, based on Wang and Lee (2009) weighting method, we extend objective weight to our proposed IT2FS entropy method to measure the objective weight in the decision matrix. The advantage of the proposed approach is that it is involve end-users into the whole decision making process. We adopted the information entropy concept to confirm the weight of evaluating attribute which can effectively balance the influence of subjective factors. The innovative approach is capable of providing a more comprehensive methodology for decision making process. Therefore, general process of this method is listed below:

Step 1: Establish a decision matrix

Establish a decision matrix for objective weight. Weight problem can be concisely expressed in matrix format as

$$\begin{matrix} & C_1 & C_2 & \cdots & C_n \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{matrix} & \begin{bmatrix} \tilde{f}_{11} & \tilde{f}_{12} & \cdots & \tilde{f}_{1n} \\ \tilde{f}_{21} & \tilde{f}_{22} & \cdots & \tilde{f}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{f}_{m1} & \tilde{f}_{m2} & \cdots & \tilde{f}_{mn} \end{bmatrix} \end{matrix} \tag{10}$$

where $\{x_1, x_2, \dots, x_m\}$ represents the alternative and C_1, C_2, \dots, C_n represents the criteria. Each entries value considered as IT2FS values, which denoted as \tilde{f}_{mn} .

Step 2: Calculate entropy value

Use IT2FS entropy formulas (Equation 4.11c) to calculate the entropy value of each IT2FS in the decision matrix. Therefore, the entropy value is represented as follows:

$$\begin{matrix} \tilde{E} = 1 / FOU(\tilde{E}) = [\underline{\tilde{E}}, \tilde{\tilde{E}}] \\ & C_1 & C_2 & \cdots & C_n \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{matrix} & \begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} & \cdots & \tilde{E}_{1n} \\ \tilde{E}_{21} & \tilde{E}_{22} & \cdots & \tilde{E}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{E}_{m1} & \tilde{E}_{m2} & \cdots & \tilde{E}_{mn} \end{bmatrix} \end{matrix} \tag{11}$$

where E_{ij} is the IT2FS entropy value of each IT2FS in the decision matrix. Therefore calculation for E_{ij} is stated as follows:

$$E(\underline{\tilde{A}}) = \frac{\sqrt{\sum_{i=1}^n [(a_{near}(x) - a_L(x))^2 + (b_{near}(x) - b_L(x))^2 + (c_{near}(x) - c_L(x))^2 + (d_{near}(x) - d_L(x))^2]}}{\sqrt{\sum_{i=1}^n [(a_{far}(x) - a_L(x))^2 + (b_{far}(x) - b_L(x))^2 + (c_{far}(x) - c_L(x))^2 + (d_{far}(x) - d_L(x))^2]}} \tag{12}$$

$$E(\tilde{\tilde{A}}) = \frac{\sqrt{\sum_{i=1}^n [(a_{near}(x) - a_U(x))^2 + (b_{near}(x) - b_U(x))^2 + (c_{near}(x) - c_U(x))^2 + (d_{near}(x) - d_U(x))^2]}}{\sqrt{\sum_{i=1}^n [(a_{far}(x) - a_U(x))^2 + (b_{far}(x) - b_U(x))^2 + (c_{far}(x) - c_U(x))^2 + (d_{far}(x) - d_U(x))^2]}} \tag{13}$$

where $E(\tilde{A}) = [E(\underline{\tilde{A}}), E(\tilde{\tilde{A}})]$ (14)

Step 3: Divide using maximal entropy value

Then, we divided all the entropy values by using the maximal entropy value and value of h_{ij} is used to represent the outcomes of maximal entropy value, and it can be defined as:

$$\tilde{h} = 1 / FOU(\tilde{h}) = [\underline{\tilde{h}}, \tilde{\tilde{h}}]$$

$$\text{where } \tilde{h}_{i1} = \left[\left(\frac{\tilde{E}_{i1}}{\max(\tilde{E}_{j1})} \right), \left(\frac{\tilde{E}_{i1}}{\max(\tilde{E}_{i1})} \right) \right] \tag{15}$$

Step 4: Weight of attributes

Finally, calculate the weight of attributes by using the weight formula. We use w_j to represent the outcome of weight value of attribute j , and it can be defined as:

$$\begin{aligned} \tilde{w}_j &= 1 / FOU(\tilde{w}_j) = [\underline{\tilde{w}}_j, \tilde{w}_j] \\ (\underline{\tilde{w}}_j, \tilde{w}_j) &= \left[\left(\frac{1 - \tilde{a}_j}{\tilde{T}} \right), \left(\frac{1 - \tilde{a}_j}{\tilde{T}} \right) \right] \end{aligned} \tag{16}$$

where $\tilde{a}_j = \sum_{d=1}^i \tilde{h}_{dj} / n$, $\tilde{a}_j = \sum_{d=1}^i \tilde{h}_{dj} / n$, $\tilde{T} = \sum_{j=1}^i \tilde{a}_j$ and $\tilde{T} = \sum_{j=1}^i \tilde{a}_j$. The \tilde{a}_j and \tilde{a}_j are represents the summation of the normalized entropy values which are corresponding to the attribute j . The \tilde{T}_j and \tilde{a}_j is the summation of \tilde{a}_j and \tilde{a}_j and n is the number of attributes. The sum of attribute weight is $\sum (\underline{\tilde{w}}_j, \tilde{w}_j) = (1, 1)$, $(\underline{\tilde{w}}_j, \tilde{w}_j) \in [0, 1]$.

NUMERICAL EXAMPLE

In this section, we give a numerical example to test the ability of proposed method to handle multiple attribute decision making problems. Example in this section refers to MADM problem used in (Chen & Lee, 2010).

Table 1 shows the linguistic terms and their corresponding type-1 fuzzy sets, respectively. Assume that there are three decision-makers D_1, D_2 and D_3 of a software company to hire a system analysis engineer and assume that there are three alternatives x_1, x_2, x_3 and five criteria “Emotional Steadiness (C_1)”, “Oral Communication Skill (C_2)”, “Personality (C_3)”, “Past Experience (C_4)” and “Self-Confidence (C_5)”. Let X be the set of alternatives, where . Assume that the three decision-makers D_1, D_2 and D_3 use the linguistic terms shown in Table 1 to represent the evaluating values of the alternatives with respect to different attributes, respectively, as shown in Table 2. Based on the interval type-2 fuzzy set representation method, the linguistic terms shown in Table 1 can be represented by interval type-2 fuzzy sets, as shown in Table 3.

Table 1: Linguistic terms for the ratings and their corresponding type-1 fuzzy sets

Linguistic Variables for the Ratings of the Vehicles	
Very Poor (VP)	(0, 0, 0, 1; 1, 1)
Poor (P)	(0, 1, 1, 3; 1, 1)
Medium Poor (MP)	(1, 3, 3, 5; 1, 1)
Medium (M)	(3, 5, 5, 7; 1, 1)
Medium Good (MG)	(5, 7, 7, 9; 1, 1)
Good (G)	(7, 9, 9, 10; 1, 1)
Very Good (VG)	(9, 10, 10, 10; 1, 1)

Source: (Chen, 2000)

Table 2: Linguistic of decision matrix

Criteria	Alternatives	Decision-Makers		
		D ₁	D ₂	D ₃
C ₁	x ₁	MG	G	MG
	x ₂	G	G	MG
	x ₃	VG	G	F
C ₂	x ₁	G	MG	F
	x ₂	VG	VG	VG
	x ₃	MG	G	VG
C ₃	x ₁	F	G	G
	x ₂	VG	VG	G
	x ₃	G	MG	VG
C ₄	x ₁	VG	G	VG
	x ₂	VG	VG	VG
	x ₃	G	VG	MG
C ₅	x ₁	F	F	F
	x ₂	VG	MG	G
	x ₃	G	G	MG

Table 3: Linguistic terms for the ratings and their corresponding type-1 fuzzy sets

Linguistic Variables for the Ratings of the Vehicles	
Very Poor (VP)	((0, 0, 0, 1; 1, 1), (0, 0, 0, 1; 1, 1))
Poor (P)	((0, 1, 1, 3; 1, 1), (0, 1, 1, 3; 1, 1))
Medium Poor (MP)	((1, 3, 3, 5; 1, 1), (1, 3, 3, 5; 1, 1))
Medium (M)	((3, 5, 5, 7; 1, 1), (3, 5, 5, 7; 1, 1))
Medium Good (MG)	((5, 7, 7, 9; 1, 1), (5, 7, 7, 9; 1, 1))
Good (G)	((7, 9, 9, 10; 1, 1), (7, 9, 9, 10; 1, 1))
Very Good (VG)	((9, 10, 10, 10; 1, 1), (9, 10, 10, 10; 1, 1))

Source: (Chen & Lee, 2010)

RESULT

The weight of attributes is calculated using the weight formula from all the steps in Sec. 4. Therefore, the result is shown as follows:

Table 4: Entropy-based weights \tilde{w}_j for each attributes

	\tilde{w}_j
C ₁	((0.1122; 1,1), (0.1122; 1,1))
C ₂	((0.2354; 1,1), (0.2354; 1,1))
C ₃	((0.2482; 1,1), (0.2482; 1,1))
C ₄	((0.0870; 1,1), (0.0870; 1,1))
C ₅	((0.32; 1,1), (0.32; 1,1))

Results of weight attributes for each criteria is archive as Table 4. Thus, the IT2FS entropy can be used for weighting attribute in MADM.

CONCLUSION

In this paper we developed interval type-2 fuzzy set entropy weight for MADM problems. The concept of evaluation of two different methods which are interval type-2 fuzzy set and entropy weight were implemented to develop a new weight of interval type-2 fuzzy set entropy weight. A simple modification on properties of entropy weight has been made into interval type-2 fuzzy sets concepts. Numerical example has been given to demonstrate the proposed method. Hence, the proposed method provides us with a useful way to handle the fuzzy multiple attribute group decision-making problems in a more flexible and more intelligent manner due to the fact that it uses interval type-2 fuzzy sets rather than traditional type-1 fuzzy sets to represent the evaluating values and the weights of attributes.

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