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Metabelian Groups of Order at Most 24

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ABSTRACT

A group G is metabelian if there exists a normal subgroup A in G such that both A and the factor group, G / A are abelian. Equivalently, G is metabelian if and only if the commutator subgroup $[G, G]$ is abelian. The main objective of this research is to determine all metabelian groups of order at most 24. In this research, some basic concepts of metabelian groups will be presented and the determinations of metabelian groups are done based on their definition and some theorems. The Groups, Algorithms and Programming (GAP) software have been used to find the multiplication table for some groups.

Keywords: Metabelian, Commutator subgroup.

INTRODUCTION

Metabelian groups are groups that are close to being abelian, in the sense that every abelian group is metabelian, but not every metabelian group is abelian. This closeness is reflected in the particular structure of their commutator subgroups. In the Russian mathematical literature, by a metabelian group one sometimes means a nilpotent group of nilpotency class two (Kurosh, 1955).

The term metabelian was earlier used for groups of nilpotency class two, but is no longer used in that sense. Sometimes, the term metabelian or derived length two or solvable length two is used specifically for a metabelian group whose derived length is precisely two, i.e., a nonabelian metabelian group. This is more restrictive than the typical usage of the term. The property of being metabelian arises by applying the meta operator to the group property of being Abelian. Equivalently metabelian can be described as Abelian-by-Abelian, where by denotes the group extension operator. A direct product of metabelian groups is metabelian.

In this research, metabelian groups of order at most 24 will be found.

SOME BASIC CONCEPTS AND PROPERTIES IN METABELIAN GROUPS

Some main definitions and theorems of metabelian groups are stated as follows :

Definition 2.1 (Wisnesky, 2005) Metabelian
A group G is metabelian if there exists a normal subgroup $A < G$ such that both A and G / A are abelian.□

Definition 2.2 (Snaith, 2003) Commutator
Given $\alpha, b \in G$.. The commutator of *a* and *b,* denoted by [*a,b*] is the element

$$[a,b] = a^{-1}b^{-1}ab \in G$$

The commutator subgroup [*G, G*] $\leq$ *G* is defined to be the smallest subgroup of *G* which contains all the commutators [*a, b*].

The following lemma and theorems have been proved by Wisnesky, 2005. The lemma is stated first to prove the theorem followed.

Lemma 2.1 (Wisnesky, 2005)
Let *G* be a group and *N* a normal subgroup of *G*. Then $gN = N$ implies $g \in N$. □

Theorem 2.1 (Wisnesky, 2005)
Let *G* be a group and *N* a normal subgroup of *G.* Then *G/N* is abelian if and only if commutator subgroup $G' = [G,G] \subseteq N$.

Proposition 2.1 (Wisnesky, 2005)
Every abelian group is metabelian.

Theorem 2.2 (Wisnesky, 2005)
G is metabelian if and only if $G'' = 1$ (G'' is the commutator subgroup of G' and G' is commutator subgroup of *G*).

Theorem 2.3 (Wisnesky, 2005)
If *H* is a subgroup of a metabelian group *G*, then is metabelian.

The following definitions and theorems will be used in proving all metabelian groups of order at most 24. Some basic concepts in group theory are stated.

Definition 2.3 (Wisnesky, 2005) Semi Direct Product
Let $N < G$ and there is a subgroup *H* such that $G = HN$ and $H \cap N = \{1\}$. Then *G* is said to be the semidirect product of *N* and *H* denoted by $G=N<H$ $G = N \rtimes H$.

Definition 2.4 (Fraleigh, 2000) Generator
An element *a* of a group *G* generates *G* and *a* is a generator for *G* if $\langle a \rangle = G$.

Definition 2.5 (Fraleigh, 2000) Cyclic Group
A group G is called cyclic if there is an element *a* in G such that $G = \{a^n / n \in Z\}$

Definition 2.6 (Fraleigh, 2000) Normal Subgroup
A subgroup *H* of a group *G* is normal if its left and right cosets coincide, that is, if $gH = Hg$ for all $g \in G$.

Definition 2.7 (Fraleigh, 2000) Cosets
Let H be a subgroup of a group G. The subset $aH = \{ah \mid h \in H\}$ of G is the left coset of H containing a, while the subset $Ha = \{ha \mid h \in H\}$ is the right coset of H containing a.

Corollary 2.1 (Fraleigh, 2000)
Let H be a normal subgroup of G. Then, the cosets H form a group G/H under the binary operation $(aH)(bH) = (ab)(H)$.

Definition 2.8 (Fraleigh, 2000) Factor Group
The group G/H in the preceding corollary is the factor group (or quotient group) of G modulo H. Recall that a subgroup H of G is normal if its left and right cosets coincide.

Definition 2.9 (Fraleigh, 2000) Order of G
If G is a finite group, then the order of G, $|G|$, is the number of elements in G. In general, for any finite set S, $|S|$, is the number of elements in S.

Definition 2.10 (Fraleigh, 2000) Presentation
Let A be a set and let $\{r_i\} \subseteq F[A]$. Let R be the least normal subgroup of $F[A]$ containing the r_i. An isomorphism φ of $F[A]/R$ onto a group G is a presentation of G. The set A and $\{r_i\}$ give a group presentation. The set A is the set of generators for the presentation and each r_i is a relator. Each $r \in R$ is consequence of $\{r_i\}$. An equation $r_i = 1$ is a relation. A finite presentation is one in which both A and $\{r_i\}$ are finite sets.

Definition 2.11 (Fraleigh, 2000) Centre of a Group G
The centre $Z(G)$ of a group G is the subset of elements in G that commute with every element of G. In symbols, $Z(G) = \{a \in G \mid ax = xa, \forall x \in G\}$.

Theorem 2.4 (Fraleigh, 2000)
The direct product of abelian group is abelian.

Theorem 2.5 (The group property wiki, 2010)
A direct product of metabelian groups is metabelian.

Theorem 2.6 (The group property wiki, 2010)

Any dihedral group is metabelian.

Theorem 2.7 (Fraleigh, 2000)
Every cyclic group is abelian.

Theorem 2.8 (Fraleigh, 2000)
A group of prime order is cyclic.

Theorem 2.9 (Fraleigh, 2000)
If the index of H in G is 2, then H is a normal subgroup. In symbols, we write:
$|G:H| = 2 \Rightarrow H \triangleleft G$.

Theorem 2.10 (Fraleigh, 2000)
The center, $Z(G)$, of a group G is always normal.

THE DETERMINATION OF METABELIAN GROUPS OF ORDER LESS THAN 24

We will show that all groups of order less than 24 are metabelian.

Proof

First we start with all abelian groups. There are 59 groups of order less than 24. Only 34 groups that are abelian (refer Table 1), thus they are metabelian by Proposition 2.

Now we consider all nonabelian groups of order less than 24. There are 25 of them (refer Table 1). We will consider each of the cases below.

Table 1: All Groups of Order Less Than 24

No	Groups	Group Order	Abelian or not	No	Groups	Group Order	Abelian or not
1	Z_1	1	Yes	31	$Z_2 \times Z_4 \times Z_2$	16	Yes
2	Z_2	2	Yes	32	$Z_2 \times Z_8$	16	Yes
3	Z_3	3	Yes	33	$Z_4 \times Z_4$	16	Yes
4	Z_4	4	Yes	34	D_8	16	No
5	$Z_2 \times Z_2$	4	Yes	35	Quasihedral-16	16	No
6	Z_5	5	Yes	36	Q_8	16	No
7	Z_6	6	Yes	37	$D_4 \times Z_2$	16	No
8	S_3	6	No	38	$Q \times Z_4$	16	No
9	Z_7	7	Yes	39	Modular -16	16	No
10	Z_8	8	Yes	40	B	16	No
11	$Z_2 \times Z_4$	8	Yes	41	K	16	No
12	$Z_2 \times Z_2 \times Z_2$	8	Yes	42	$G_{4,4}$	16	No
13	D_4	8	No	43	Z_{17}	17	Yes

Table 1: continued

14	$Q = Quarternion$	8	No	44	Z_{18}	18	Yes
15	Z_9	9	Yes	45	$Z_3 \times Z_6$	18	Yes
16	$Z_3 \times Z_3$	9	Yes	46	D_8	18	No
17	Z_{10}	10	Yes	47	$S_3 \times Z_3$	18	No
18	D_5	10	No	48	$(Z_3 \times Z_3) \rtimes Z_2$	18	No
19	Z_{11}	11	Yes	49	Z_{19}	19	Yes
20	Z_{12}	12	Yes	50	Z_{20}	20	Yes
21	$Z_2 \times Z_6$	12	Yes	51	$Z_2 \times Z_{10}$	20	Yes
22	$T = Z_3 \rtimes Z_4$	12	No	52	D_{10}	20	No
23	A_4	12	No	53	$Fr_{20} \cong Z_5 \rtimes Z_4$	20	No
24	D_6	12	No	54	$Z_4 \rtimes Z_5$	20	No
25	Z_{13}	13	Yes	55	Z_{21}	21	Yes
26	Z_{14}	14	Yes	56	$Fr_{21} \cong Z_7 \rtimes Z_3$	21	No
27	D_7	14	No	57	Z_{22}	22	Yes
28	Z_{15}	15	Yes	58	D_{11}	22	No
29	Z_{16}	16	Yes	59	Z_{23}	23	Yes
30	$Z_2 \times Z_2 \times Z_2 \times Z_2$	16	Yes				

Theorem 3.1

S_3 is metabelian.

Proof: S_3 has six elements which are $\{(1), (12), (13), (23), (123), (132)\}$. Let A_3 be the alternating subgroup of S_3 with elements $\{(1), (123), (132)\}$. A_3 is cyclic since the order is prime (Theorem 2.8). Thus A_3 is abelian (Theorem 2.7). A_3 is a normal subgroup of S_3 since it has index two (Theorem 2.9) which is $\left| S_3 / A_3 \right| = 2$. The factor group of S_3 / A_3 is abelian since $S_3 / A_3 \cong \mathbb{Z}_2$ Thus, by definition, S_3 is metabelian. □

Theorem 3.2
By Theorem 2.6, any dihedral groups are metabelian. Therefore, all dihedral groups of order less than 24 such as $D_3 D_4, D_5, D_6, D_7, D_8, D_9, D_{10}$ and D_{11} are all metabelian.

Theorem 3.3
$Q = \langle a, b \mid a^4 = 1, a^2 = b^2, aba = b \rangle$, the quaternion group of order eight is metabelian.
Proof: The elements of Q can be written as $Q = \{1, -1, i, -i, j, -j, k, -k\}$. Let $A = \langle -1 \rangle = \{-1, 1\}$and A is the center of a group Q. Then $A=Z(G)$ is normal in Q (Theorem 2.10). is also abelian since it is cyclic. Furthermore, the order of factor group $\left| Q/A \right| = |Q|/|A| = 8/2 = 4$. Hence $Q/A \cong \mathbb{Z}_4 \; or \; \mathbb{Z}_2 \times \mathbb{Z}_2$ Thus the factor group, Q/A is abelian. Hence, is metabelian.

Theorem 3.4
$T = Z_3 \rtimes Z4 = \langle a, b \mid a^4 = b^3 = 1, aba = a \rangle$, the semidirect product of a cyclic group of order three with a cyclic group of order four is metabelian.
Proof: Since T is semidirect product, then there exist one normal subgroup, that is $\mathbb{Z}_3 \triangleleft T$ (Definition 2.3). Let $A = \mathbb{Z}_3$. Then, $|A| = 3$ and A is abelian since the order is prime. The order of factor group, $\left| T/A \right| = |T|/|A| = 12/3 = 4$. Hence, $T/A \cong \mathbb{Z}_4 \; or \; \mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, the factor group T/A is abelian. Therefore is metabelian.

Theorem 3.5
$A_4 = \langle a, b, c \mid a^2 = b^2 = c^3 = 1, ba = ab, ca = abc, cb = ac \rangle$, the alternating group of order 12 is metabelian.
Proof: Let $A \leq A_4$. The elements of A can be written as
A = {(1), (12), (34), (13), (24), (14), (23)} and $|A| = 4$. Now, we have the elements of
A_4 = {(1), (12), (34), (13), (24), (14), (23), (123), (132), (234), (243), (134), (143), (124), (142)}.

Then, we have the left cosets and the right cosets as follows:
(1) A =A = {(1), (12), (34), (13), (24), (14), (23)}
(123) A = {(123), (134), (243), (142)} = A (123)}
(132) A = {(132), (234), (143), (124) = A (23)}

Since the left cosets and the right cosets are same, then $A \triangleleft A_4$ and A is also abelian since its commute with all of its elements, i.e $ab = ba$, for all a, b ∈ A. Furthermore, the order of factor group $\left| A_4/A \right| = |A_4|/|A| = 12/4 = 3$. Hence, $A_4/A \cong \mathbb{Z}_3$ thus A_4/A is abelian. Therefore, A_4 is metabelian.

Theorem 3.6
Quasihedral-16 = $\langle a, b \mid a^8 = b^2 = 1, bab = a^3 \rangle$ is metabelian.
Proof: Let G = Quasihedral-16 and A= $\langle \alpha \rangle$. Then, $|A| = 8$ since $\alpha^8 = 1$. Furthermore A is cyclic thus A is abelian. Next, $A \triangleleft G$ since the index of A in G is 2 (Theorem 2.9). That is, $\left| G/A \right| = |G|/|A| = 16/8 = 2$. Furthermore, $G/A \cong \mathbb{Z}_2$. Hence, G/A is abelian. Therefore, Quasihedral-16 is metabelian.

Theorem 3.7

$Q_8 = \langle a, b \mid a^8 = 1, a^4 = b^2, aba = b \rangle$, the quarternion group of order 16 is metabelian.

Proof: Let A = $\langle a \rangle$. Then $|A| = 8$ since $a^8 = 1$. A is cyclic thus A is abelian. Next, $A \triangleleft Q_s$ since the index is 2 (Theorem 2.9). Furthermore, $\left| Q_s/A \right| = |Q_s|/|A| = 16/8 = 2$. Hence, $Q_s/A \cong \mathbb{Z}_2$. Thus Q_s/A is abelian. Therefore, is metabelian.

Theorem 3.8

By Theorem 2.5, the direct product of metabelian groups is metabelian. Therefore, $D_4 \times \mathbb{Z}_2 Q \times \mathbb{Z}_2$ and $S_3 \times \mathbb{Z}_3$ are metabelian.

Theorem 3.9

Modular -16 = $\langle a, b \mid a^8 = b^2 = 1, ab = ba^5 \rangle$is metabelian.

Proof: Let G = Modular -16 and $A = \langle a \rangle$. Then, $|A| = 8$ since $a^8 = 1$ and A is cyclic thus A is abelian. Next, $A \triangleleft G$ since the index is 2 (Theorem 2.9). Furthermore, the order of factor group, $\left| G/A \right| = |G|/|A| = 16/8 = 2$. Hence, $G/A \cong \mathbb{Z}_2$ and the factor group G/A is abelian. Therefore, is metabelian.

Theorem 3.10

$\langle a, b \mid a^4 = b^4 = 1, ab = ba3 \rangle$ is metabelian.

Proof: Let $A = \langle a^2 \rangle = \{a^2, e\}$. Then, $|A| = 2$ since $a^4 = 1$. A is cyclic. Thus A is abelian. Next, we find the left cosets and the right cosets of B. Let the elements of B are:

$$B = \{e, a, a^2, a^3, b, b^2, b^3, ab, ab^2, ab^3, a^2b, a^2b^2, a^2b^3, a^3b, a^3b^2, a^3b^3\}$$

Then, we have the left cosets and right cosets as follows:

$$eA = \{a^2, e\} = A = Ae$$

$$aA = \{a^3, e\} = Aa$$

$$bA = \{a^2, \text{b}, \text{b}\} = Ab$$

$$b^2A = \{a^2, b^2, b^2\} = Ab^2$$

$$b^3A = \{a^2, b^2, b^3\} = Ab^3$$

$$abA = \{a^3, b, ab\} = Aab$$

$$ab^2A = \{a^3, b^2, ab^2\} = Aab^2$$

$$ab^3A = \{a^3, b^3, ab^3\} = Aab^3$$

Since the left cosets = the right cosets, then $A \triangleleft B$. Furthermore, the factor group B/A = {*A, aA, bA, b^2A, b^3A, abA, ab^2A, ab^3A*}. Then, the order of factor group $\left| B/A \right| = |B|/|A| = 16/8 = 8$. B/A is abelian since for all . Therefore,is metabelian.

Theorem 3.11
$K = \langle a,b,c \mid a^4 = b^2 = c^2 = 1, cbca^2b = 1, bab = a, cac = a \rangle$ is metabelian.
Proof: Let A = $\langle \alpha \rangle$= {*e, a, a, a^2, a^3*}. Then, |A} = 4 since a^4=1. is cyclic. Thus is abelian. Next, we find the left cosets and the right cosets of *K*. Let the elements of *K* are:

$$K = \{e,a,a^2,a^3,b,c,ab,ac,bc,a^2b,a^2c,a^3b,a^3c,abc,a^2bc,a^3bc\}$$

Then, we have the left cosets and right cosets as follows:

$$eA = \{e,a,a^2,a^3\} = A = Ae$$

$$bA = \{b,a^2b,a^3\mathrm{b}\} = Ab$$

$$cA = \{\mathrm{c},ac,a^2c^2,a^3c\} = Ac$$

$$bcA = \{\mathrm{bc},abc,a^2,\mathrm{bc},a^3bc\} = Abc$$

Since the left cosets = the right cosets, then $A \triangleleft K$. Furthermore, the order of factor group $\left| K/A \right| = |K|/|A| = 16/8 = 4$. Hence, $K/A \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and K/A is abelian. Therefore *K*,is metabelian.

Theorem 3.12
$G_{4,4} = \langle a,b \mid a^4 = b^4 = 1, abab = 1, ba^3 = ab^3 \rangle$ is metabelian.
Proof: Let $A = \langle a^2b^2 \rangle = \{e,a^2b^2\}$. Then, $|A| = 2$. A is cyclic. Thus A is abelian. Next, we find the left cosets and the right cosets of $G_{4,4}$. Let the elements of $G_{4,4}$ are:

$$G_{4,4} = \{e,a,a^2,a^3,b,b^2,ab,ab^2,a^2b,a^2b^2,a^3b,a^3b^2,aba,a^2ba,a^3ba,ba\}$$

Then, we have the left cosets and right cosets as follows:

$$eA = \{e,a^2,b^2\} = A = Ae$$

$$aA = \{\mathrm{a},a^3,b^2\} = Aa$$

$$a^2A = \{a^2,b^2\} = Aa^2$$

$$a^3A = \{a^3,ab^2\} = Aa^3$$

$$bA = \{b,a^3,ba\} = Aa^3$$

$$abA = \{ab,ba\} = Aab$$

$a^2bA = \{a^2b, aba\} = Aa^2b$

$a^3bA = \{a^3b, a^2ba\} = Aa^3b$

Since the left cosets = the right cosets, then $A \triangleleft G_{44}$. Furthermore, the order of factor group $\left|G_{4,4}/A\right| = |G_{4,4}|/|A| = 16/4 = 4$. . Hence, $G_{4,4}/A \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the factor group $G_{4,4}/A$ is abelian. Therefore, $G_{4,4}$ is metabelian.

Theorem 3.13

$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 = \langle a,b,c \mid a^2 = b^3 = c^3 = 1, bc = cb, bab = a, cac = a \rangle$, the semidirect product of two direct product of cyclic group of order three with the cyclic group of order two is metabelian.

Proof: Let $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ and $A = (\mathbb{Z}_3 \times \mathbb{Z}_3)$. Then $A \triangleleft G$ since the index is 2 (Theorem 2.9) and is abelian since the direct product of abelian group (Theorem 2.4). Next, the order of factor group, $\left|G/A\right| = |G|/|A| = 18/9 = 2$.. Hence, $G/A \cong \mathbb{Z}_2$ and the factor group, G/A is abelian. Therefore, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 =$ is metabelian.

Theorem 3.14

$Fr_{20} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4 = \langle a,b \mid a^4 = b^5 = 1, ba = ab^2 \rangle$, the frobenius group of order 20 isomorphic to semidirect product of cyclic group of order five with the cyclic group of order four is metabelian.

Proof: Let $G = Fr_{20} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$ and $A = \mathbb{Z}_5 = \langle b \rangle$ Then, $|A| = 5$ since $b^5 = 1$. A is cyclic thus, A is abelian. $A \triangleleft G$ since it is the semidirect product of G (Definition 2.3). Furthermore, the order of factor group, $\left|G/A\right| = |G|/|A| = 20/5 = 4$.. Hence, $G/A \cong (\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the factor group, G/A is abelian. Therefore, Fr_{20} is metabelian.

Theorem 3.15

$\mathbb{Z}_4 \rtimes \mathbb{Z}_5 = \langle a,b \mid a^4 = b^5 = 1, ba = ab^2 \rangle$, the semidirect product of cyclic group of order four with the cyclic group of order 5 is metabelian.

Proof: Let $G = G = \mathbb{Z}_4 \rtimes \mathbb{Z}_5$ and $A = \mathbb{Z}_4$. Then, $|A| = 4$ and A is cyclic thus, is abelian. Next, $A \triangleleft G$ (Definition 2.3). Furthermore, the order of factor group, $\left|G/A\right| = |G|/|A| = 20/4 = 5$. Hence, $G/A \cong \mathbb{Z}_5$ and the factor group, G/A is abelian. Therefore, $\mathbb{Z}_4 \rtimes \mathbb{Z}_5$ is metabelian.

Theorem 3.16

$Fr_{21} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3 = \langle a,b \mid a^3 = b^7 = 1, ba = ab^2 \rangle$, the frobenius group of order 21 isomorphic to semidirect product of cyclic group of order seven with the cyclic group of order three is metabelian. ▪

Proof: Let $G = Fr_{21} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ and $A = \mathbb{Z}_7 = \langle b \rangle$. Then, $|A| = 7$ since $b^7 = 1$. A is cyclic thus, A is abelian. $A \triangleleft G$ since it is the semidirect product of G (Definition 2.3). Furthermore, the order of factor group, $\left| G/A \right| = |G|/|A| = 21/7 = 3$. Hence, $G/A \cong \mathbb{Z}_3$ and the factor group, G/A is abelian. Therefore,is metabelian.■

In this section, we conclude that all groups of order less than 24 are metabelian. ■

THE DETERMINATION OF METABELIAN GROUPS OF ORDER 24.

Now, we will prove for all group of order 24 whether it is metabelian or not. In Table 4.1, there are 15 groups of order 24.

Table 42: All Groups of Order 24

No	Groups	Group Order	Abelian or not	No	Groups	Group Order	Abelian or not
1	Z_{24}	24	Yes	9	$A_4 \times Z_2$	24	No
2	$Z_2 \times Z_{12}$	24	Yes	10	Q_{12}	24	No
3	$Z_2 \times Z_2 \times Z_6$	24	Yes	11	$Sl(2,3)$	24	No
4	S_4	24	No	12	D_{12}	24	No
5	$S_3 \times Z_4$	24	No	13	$M = Z_2 \times (Z_3 \rtimes Z_4)$	24	No
6	$S_3 \times Z_2 \times Z_2$	24	No	14	$N = (Z_3 \rtimes Z_8)$	24	No
7	$D_4 \times Z_3$	24	No	15	$Z_3 \rtimes Q$	24	No
8	$Q \times Z_3$	24	No				

Using same method as above, we can conclude that all three abelian groups, groups of direct product and dihedral groups are metabelian which are,

$\mathbb{Z}_{24}, \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6, S_3 \times \mathbb{Z}_4, S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_4 \times \mathbb{Z}_3, Q \times \mathbb{Z}_3 A_4 \times \mathbb{Z}_2$ and $D_{12.}$

Theorem 4.1
$S_4 = \langle a,b,c \mid a^2 = b^2 = c^2 = 1, (ab)^3 = (bc)^3 = (ac)^3 = 1 \rangle$, the symmetric group of order 24 is metabelian.

Proof: We have,
S_4= {(1), (12), (13), (14), (23), (24), (34), (12), (34), (13), (234), (14), (23), (123), (132), (234), (243) (134),)143), (124), (142), (1234), (1243), (1324), (1342), (1423), (1432)

Normal subgroups of S_4 are (1), A_4 and A. (1) is normal since it is the trivial subgroup. The trivial subgroup is always normal and abelian but the factor group $S_4/(1)$ is not abelian. Next, A_4 is normal since the index is 2, but A_4 is not abelian. Now, will prove the normality of A in S_4. Let the elements of $A = \{(1), (12), (34), (13), (24), (14), (23)\}$. Then, the factor group of S_4 is

$$S_4/A = \{(1),(12),(13),(14),(23),(24),(34),...\}/\{(1),(12),(34),(13),(24),(314),(23)\}$$

Then we have, the left and the right cosets as follows:

(1) A =A = {(1), (12), (34), (13), (24), (14), (23)} = A(1)

(12) A = {(12), (34), (1423), (1324)} = A (12)

(13) A = {(13), (1432), (24), (1234) = A (13)}

(14) A = {(14), (1342), (31243), (23)} = A (14)

(123) A = {(123), (243), (142), (134)} = A (123)

(124) A = {(124), (234), (143), (132) = A (124)}

Since the left and the right cosets are same, hence $A \triangleleft S_4$, A is also abelian since its commute with all of its elements,i.e $ab = ba$, for all a, $b \in A$. But the factor group of S_4 is not abelian since, for example:

since, for example:

{(12)A}, {(13) A}= (12) (13)A = (132)A $\neq$ (123)A = (13) (12) A = {(13) A} {(12)A}

Hence S_4 is not metabelian.

Theorem 4.2
$Q_{12} = \langle a,b,c \mid a^2 = b^6 = c^{12} = 1, bab = a \rangle$, quartenion group of order 24 is metabelian.

Proof: Let $A = \langle c \rangle$*. Then,* $|A| = 12$ since $c^{12} = 1$. A is cyclic thus A is abelian. Next, $A \triangleleft Q_{12}$ since the index is 2 (Theorem 2.9). Furthermore, the order of factor group, $\left| Q_{12}/A \right| = |Q12| / |A| = 24/12 = 2$. Hence, $\left| Q_{12}/A \right| \cong \mathbb{Z}_2$ and the factor group, Q_{12}/A is abelian. Therefore, is metabelian.■

Theorem 4.3
$Sl(2,3) = \langle a,b,c \mid a^4 = c^3 = 1, a^2 = b^2, aba = b, ac = cb, cab = bc \rangle$ is metabelian.
Proof: First we need to find normal subgroup of *Sl*(2,3) then show it metabelian or not. Suppose $A \leq Sl(2,3)$.

a) Let $A = \langle e \rangle = \{e\}$ is trivial subgroup. *A* is normal subgroup of *A* since the trivial subgroups is always normal.

b) Let $A = \langle e \rangle = \{e\}$. Then, we have the left cosets and the right cosets are as follows:
$eA = \{e,a\} = Ae = A$
$bA = \{b,ac\} \neq \{b,ab\} = Ab$

Since there exist the left cosets not equal to right cosets, therefore $A = \langle a \rangle = \{e, a\}$ is not normal in *Sl*(2,3).

c) Let $A = \langle b \rangle = \{e,b,d,bd\}$. Then, we have the left cosets and the right cosets are as follows:
$eA = \{e,b,d,bd\} = Ae = A$
$aA = \{a,ab,ad,abd\} \neq \{a,ac,ad,acd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle b \rangle = \{e,b,d,bd\}$ is not normal in *Sl*(2,3).

d) $A = \langle c \rangle = \{e,c,d,cd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e,c,d,cd\} = Ae = A$
$aA = \{a,ac,ad,acd\} \neq \{a,abc,ad,abcd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle c \rangle = \{e,c,d,cd\}$ is not normal in *Sl*(2,3).

e) $A = \langle d \rangle = \{e,d,\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e,d,\} = Ae = A$
$aA = \{a,ad\} = Aa$
$bA = \{b,bd\} = Ab$
$cA = \{c,cd\} = Ac$
$a^2A = \{a^2,a^2d\} = Aa^2$
$abA = \{ab,abd\} = Aab$
$acA = \{ac,acd\} = Aac$
$bcA = \{bc,bcd\} = Abc$
$a^2bA = \{a^2b,a^2bd\} = Aa^2b$
$a^2cA = \{a^2c,a^2cd\} = Aa^2c$
$abcA = \{abc,abcd\} = Aabc$
Since the left cosets equal to right cosets, therefore $A = \langle c \rangle = \{e,c,d,cd\}$ is normal in *Sl*(2,3).

f) $A = \langle a^2 \rangle = \{e,a,a^2\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e,a,a^2\} = Ae = A$
$bA = \{b,ac,a^2bc\} \neq \{b,ab,a^2b\} = Ab$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle a^2 \rangle = \{e,a,a^2\}$ is not normal in *Sl*(2,3).

g) $A = \langle ab \rangle = \{e, ab, a^2bcd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, ab, a^2bcd\} = Ae$
$aA = \{a, a^2b, bcd\} \neq \{a, a^2c, bd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle ab \rangle = \{e, ab, a^2bcd\}$ is not normal in *Sl*(2,3).

h) $A = \langle ac \rangle = \{e, ac, a^2bd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, ac, a^2bd\} = Ae = A$
$aA = \{a, a^2c, bd\} \neq \{a, a^2bc, cd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle ac \rangle = \{e, ac, a^2bd\}$ is not normal in *Sl*(2,3).

i) $A = \langle ad \rangle = \{e, a, d, a^2, ad, a^2d\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, a, d, a^2, ad, a^2d\} = Ae = A$
$bA = \{b, ac, bd, a^2bc, acd, a^2bcd\} \neq \{b, ab, bd, a^2b, abd, a^2bd\} = Ab$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle ad \rangle = \{e, a, d, a^2, ad, a^2d\}$ is not normal in *Sl*(2,3).

j) $A = \langle bc \rangle = \{e, d, bc, bcd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, d, bc, bcd\} = Ae = A$
$aA = \{a, ad, abc, abcd\} \neq \{a, ad, ab, abd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle bc \rangle = \{e, d, bc, bcd\}$ is not normal in *Sl*(2,3).

k) $A = \langle bd \rangle = \{e, b, d, bd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, b, d, bd\} = Ae = A$
$aA = \{a, ab, ad, abd\} \neq \{a, ac, ad, acd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle bd \rangle = \{e, b, d, bd\}$ is not normal in *Sl*(2,3).

l) $A = \langle cd \rangle = \{e, c, d, cd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, c, d, bcd\} = Ae = A$
$aA = \{a, ac, ad, acd\} \neq \{a, abc, ad, abcd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle cd \rangle = \{e, c, d, cd\}$ is not normal in *Sl*(2,3).

m) $A = \langle a^2b \rangle = \{e, d, ac, a^2b, acd, a^2bd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, d, ac, a^2b, acd, a^2bd\} = Ae = A$
$aA = \{a, ad, a^2c, b, a^2cd, bd\} \neq \{a, ad, a^2bc, c, a^2bcd, cd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle a^2b \rangle = \{e, d, ac, a^2b, acd, a^2bd\}$ is not normal in *Sl*(2,3).

n) $A = \langle a^2c \rangle = \{e, d, a^2c, abc, a^2cd, abcd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, d, a^2c, abc, a^2cd, abcd\} = Ae = A$
$aA = \{a, ad, c, a^2bc, cd, a^2bcd\} \neq \{a, ad, bc, a^2b, bcd, a^2bd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle a^2c \rangle = \{e, d, a^2c, abc, a^2cd, abcd\}$ is not normal in *Sl*(2,3).

o) $A = \langle a^2d \rangle = \{e, a, d, ad, a^2, a^2d\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, a, d, ad, a^2, a^2d\} = Ae = A$
$bA = \{b, ac, bd, a^2bc, acd, a^2bcd\} \neq \{b, ab, bd, abd, a^2b, a^2bd\} = Ab$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle a^2d \rangle = \{e, a, d, ad, a^2, a^2d\}$ is not normal in *Sl*(2,3).

p) $A = \langle abc \rangle = \{e, abc, a^2cd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, abc, a^2cd\} = Ae = A$
$aA = \{a, a^2bc, cd\} \neq \{a, a^2b, bcd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle abc \rangle = \{e, abc, a^2cd\}$ is not normal in *Sl*(2,3).

q) $A = \langle abd \rangle = \{e, d, ab, abd, a^2bc, a^2bcd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, d, ab, abd, a^2bc, a^2bcd\} = Ae = A$
$aA = \{a, ad, a^2b, a^2bd, bc, bcd\} \neq \{a, ad, a^2c, a^2cd, b, bd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle abd \rangle = \{e, d, ab, abd, a^2bc, a^2bcd\}$ is not normal in *Sl*(2,3).

r) $A = \langle acd \rangle = \{e, d, ac, acd, a^2b, a^2bd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, d, ac, acd, a^2b, a^2bd\} = Ae = A$
$aA = \{a, ad, a^2c, a^2cd, b, bd\} \neq \{a, ad, a^2bcd, c, cd, a^2bc\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle acd \rangle = \{e, d, ac, acd, a^2b, a^2bd\}$ is not normal in *Sl*(2,3).

s) $A = \langle bcd \rangle = \{e, d, bc, bcd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, d, bc, bcd\} = Ae = A$
$aA = \{a, ad, abc, abcd\} \neq \{a, ad, ab, abd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle bcd \rangle = \{e, d, bc, bcd\}$ is not normal in *Sl*(2,3).

t) $A = \langle a^2bc \rangle = \{e, d, ab, abd, a^2bc, a^2bcd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, d, ab, abd, a^2bc, a^2bcd\} = Ae = A$
$aA = \{a, ad, a^2b, a^2bd, bc, bcd\} \neq \{a, ad, a^2c, a^2cd, b, bd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle a^2bc \rangle = \{e, d, ab, abd, a^2bc, a^2bcd\}$ is not normal in *Sl*(2,3).

u) $A = \langle a^2bd \rangle = \{e, ac, a^2bd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, ac, a^2bd\} = Ae = A$
$aA = \{a, a^2c, bd\} \neq \{a, a^2bc, cd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle a^2bd \rangle = \{e, ac, a^2bd\}$ is not normal in $Sl(2,3)$.

v) $A = \langle a^2cd \rangle = \{e, abc, a^2cd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, abc, a^2cd\} = Ae = A$
$aA = \{a, a^2bc, cd\} \neq \{a, a^2b, bcd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle a^2cd \rangle = \{e, abc, a^2cd\}$ is not normal in $Sl(2,3)$.

w) $A = \langle abcd \rangle = \{e, d, a^2c, abc, a^2cd, abcd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, d, a^2c, abc, a^2cd, abcd\} = Ae = A$
$aA = \{a, ad, c, a^2bc, cd, a^2bcd\} \neq \{a, ad, bc, a^2b, bcd, a^2bd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle abcd \rangle = \{e, d, a^2c, abc, a^2cd, abcd\}$ is not normal in $Sl(2,3)$.

x) $A = \langle a^2bcd \rangle = \{e, ab, a^2bcd\}$
Then, we have the left cosets and the right cosets are as follows:
$eA = \{e, ab, a^2bcd\} = Ae = A$
$aA = \{a, a^2b, bcd\} \neq \{a, a^2c, bd\} = Aa$
Since there exist the left cosets not equal to right cosets, therefore $A = \langle a^2bcd \rangle = \{e, ab, a^2bcd\}$ is not normal in $Sl(2,3)$.

Since we have two normal subgroup of $Sl(2,3)$ which is the trivial subgroup, $\langle e \rangle$ and $\langle d \rangle = \{e, d\}$ now we show whether $Sl(2,3)$ is metabelian or not. First let $A = \langle e \rangle$ is normal subgroup. A is cyclic. Thus A is abelian. But the factor group $Sl(2,3)/A$ is not abelian since $Sl(2,3)/A = Sl(2,3)$ which is not abelian. Next, let $B = \langle d \rangle = \{e, d,\}$ be normal subgroup. B is cyclic. Thus B is abelian. But the factor group $Sl(2,3)/B$ is not abelian since $(aB)(bB) = (ab)B = \{ab, abd\} \neq \{ac, acd\}(ac)B = (bB)(aB)$. Hence $Sl(2,3)$ is not metabelian.

Theorem 4.4
$M = \mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4 \langle a, b, c, d \mid a^4 = b^6 = 1, bab = a \rangle$ is metabelian.
Proof: Let $M = \mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ and $A = \langle cd \rangle$. The elements of M and A can be written as follows:
$M = \{e, a, b, c, d, ab, ac, ad, bc, bd, cd, d^2, abc, abd, acd^2\}$
$ad^2, bcd, bd^2, cd^2, abcd. abd^2, acd^2, bcd^2, abcd^2\}$
$A = \{e, c, d, cd, d^2, cd^2\}$.

Next, we find the left and the right cosets of M/A, which are,
$A = \langle cd \rangle = \{e, c, d, cd, d^2, cd^2\} = A$
$aA = \{a, ac, ad, acd, ad^2, acd^2\} = Aa$
$bA = \{b, bc, bd, bcd, bd^2, bcd^2\} = Ab$
$abA = \{ab, abc, abd, abcd, abd^2, abcd^2\} = Aab$

Therefore, we can see that $A \triangleleft M$ since left cosets = right cosets. $A = \langle cd \rangle$ is cyclic thus is abelian. Furthermore, $|M| = 24$ and $|A| = 6$. Then, the order of factor group, $\left| M/A \right| = |M|/|A| = 24/6 = 4$ Hence, $M/A \cong \mathbb{Z}_4$ or $\mathbb{Z}_2$ and the factor group, M/A is abelian. Therefore, M is metabelian.

Theorem 4.5

$N = (\mathbb{Z}_3 \rtimes \mathbb{Z}_8) = \langle a, b, c, d \mid a^3 = b^4 = c^2 = 1, bcb = c, aba = b, ac = ca \rangle$ is metabelian.

Proof: Let $A = \langle bd \rangle$ and the elements of can be written as $A = \langle bd \rangle = \{bd, cd^2, bc, d, bd^2, c, bcd, d^2, b, cd, bcd^2, e\}$. Then, $|A| = 12$ and A is cyclic. Thus A is abelian. $A \triangleleft N$ since the index is two (Theorem 2.9). Furthermore $\left| N/A \right| = |N|/|A| = 24/12 = 2$, Hence, $N/A \cong \mathbb{Z}_2$ and the factor group, N/A is abelian Therefore, N is metabelian ▪.

Theorem 4.6

$\mathbb{Z}_3 \rtimes Q = \langle a, b, c, d \mid a^2 = b^6 = c^2 = 1, ab = ba, ac = ca, cbcb = 1 \rangle$ is metabelian.

Proof: Let $G = \mathbb{Z}_3 \rtimes Q$ and $A = \langle bd \rangle$ and the elements of can be written as $A = \langle bd \rangle = \{bd, cd^2, bc, d, bd^2, c, bcd, d^2 b, cd, bcd^2, e\}$. Then, $|A| = 12$ and A is cyclic thus A is abelian. $A \triangleleft G$ since the index is two (Theorem 2.9). Furthermore, $\left| G/A \right| = |G|/|A| = 24/12 = 2$ Hence, $G/A \cong \mathbb{Z}_2$ and the factor group, G/A is abelian. Therefore, $\mathbb{Z}_3 \rtimes Q$ is metabelian.

Then, we conclude that there exist two groups of order 24 that is not metabelian which is S_4 and $Sl(2,3)$

CONCLUSIONS

With the scope of this research, 72 from 74 groups of order at most 24 are detected as metabelian groups and the rest two groups of order 24 are not metabelian which are

i) $S_4 = < a,b,c \mid a^2 = b^2 = c^2 = 1, (ab)^3 = (bc)^3 = (ac)^3 = 1 >$ and

ii) $Sl(2,3) = \langle a,b,c \mid a^4 = c^3 = 1, a^2 = b^2, aba = b, ac = cb, cab = bc \rangle$

All groups of order at most 24 have been proved as metabelian groups using their group presentations. The Groups, Algorithms and Programming (GAP) software has been used to facilitate some of the computations and proofs.

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