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Properties and Solutions of Magic Squares

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ABSTRACT

Magic square is one of the different arrays in magic number arrangements. Many studies on magic square have been carried out after the first pattern of magic square has been developed. Different magic squares have been calculated manually until 1973. Magic square is a branch of mathematical recreation combinatorics. It also has applications in Karnaugh map and contributes in number theory. This research focuses on normal magic squares of order 3, 4 and 6. Some operations and properties for magic square such as complement, rotation and reflection are discussed. Among the methods to obtain the solution for normal magic square of odd order, single-even order and double-even order are De la Laubera's method, Devedec's method and Modified De la Laubera's method respectively. In this research, solutions for normal magic squares of order 3, 4, and 6 are also generated using MATLAB programming software.

Keywords: magic square; complement; rotation; reflection; MATLAB.

ABSTRAK

Segiempat sama ajaib adalah salah satu daripada susunan-susunan pengaturan nombor ajaib. Banyak penyelidikan dalam segiempat sama ajaib telah dijalankan selepas susunan pertama segiempat sama ajaib telah dibangunkan. Pelbagai segiempat sama ajaib telah diselesaikan secara manual sehingga 1973. Segiempat sama ajaib merupakan satu cabang kombinatorik matematik rekreasi. Pengaturan ini juga mempunyai aplikasi dalam peta Karnaugh serta menyumbang kepada teori nombor. Kajian ini menumpu kepada segiempat sama ajaib biasa berperingkat 3, 4 dan 6. Beberapa operasi dan sifat–sifat segiempat sama ajaib iaitu pelengkap, putaran dan refleksi turut dibincangkan. Antara kaedah-kaedah penyelesaian segiempat sama ajaib biasa berperingkat ganjil, genap-tunggal dan genap-berpasangan adalah kaedah De la Laubera, kaedah Devedec dan kaedah Modified De la Laubera masing-masing. Justeru dalam kajian ini, penyelesaian bagi segiempat sama ajaib biasa berperingkat 3, 4 dan 6 juga dijanakan dengan menggunakan perisian pengaturcaraan MATLAB.

Kata kunci: Segiempat sama ajaib; pelengkap; putaran; refleksi; MATLAB.

INTRODUCTION

Magic squares have long been introduced as mathematical recreation to provide entertainment and interesting outlet for creating mathematical knowledge. An *n*th-order magic square is a square array of n^2 distinct integers in which the sum of the n numbers in each row, column, and diagonal is the same [1].

A normal magic square is a group of consecutive numbers starting from 1 being arranged into the shape of a square. The numbers in magic square are arranged in which each horizontal row, vertical column and its diagonal are equal to a constant. Such constant is called the magic constant. The view of magic square of order 3 is shown in Figure 1.

A	*B*	*C*
D	*E*	*F*
G	*H*	*I*

Figure 1: View of magic square of order 3,

From Figure 1, it is obtained that,

$$A + B + C = N, \qquad D + E + F = N,$$
$$G + H + I = N, \qquad A + D + G = N,$$
$$B + E + H = N, \qquad C + F + I = N,$$
$$A + E + I = N, \quad \text{and} \quad C + E + G = N.$$

Such number N is called the magic constant for magic square.

For a normal magic square of order 3, nine consecutive integers are to be arranged into each cell of magic square of order 3. The magic constant for normal magic square is equal to $(m^3 + m)/2$, which is equal to total of 15, as explained in the following.

From the eight equations above, it is obtained that,

$$2 \times (A + B + C + D + E + F + G + H + I) + (A + E + I) + (C + E + G) = 8N.$$

Since $A + B + C + D + E + F + G + H + I = 45$, thus,

$$8N = 2(45) + 2N$$
$$N = 15.$$

Therefore, the magic constant for magic square of order 3 is 15.

Below are some definitions that used in the discussion of magic squares.

Definition 1: [1] (Magic Constant (*N*))
The sum produced by each row, column, and main diagonal (and possibly other arrangements) of the magic square is called **magic constant (*N*)**.

The constant (N) of a normal magic square is $(m^3 + m) / 2$, where m is the order of magic square. However if the magic square consists of consecutive numbers but not starting at 1, the constant is $(m^3 + m) / 2 + m (a – 1)$, where a equals the starting number and m is the order.

There are some other terms used in magic arrangement. For instance, the definition of array and cell are given in Definition 2 and Definition 3 respectively.

Definition 2: [2] (Array)
An **array** is an orderly arrangement of a set of cardinal numbers, algebraic symbols, or other elements into rows, columns, files, or any other lines.

Definition 3: [3] (Cell)
Cell is the basic element of a magic square, magic cube, magic star, etc.

Each cell contains one number, usually an integer. However, it can hold a symbol or the coordinates of its location. For example there are m^2 cells in a magic square of order m; $2n$ cells in a normal magic star of order n.

There are different types of magic squares as mentioned in Definition 4 and Definition 5.

Definition 4: [4] (Magic Square of Order *m*)
A **magic square of order *m*** is an arrangement of m^2 number in a square pattern with the property that the sum of every row and column, as well as both diagonals, is the same number.

Definition 5: [5] (Normal Magic Square)
A **normal magic square** is an m-order magic square which contains the consecutive integers 1 to m^2.

In the following sections, analyses to some operation and properties for magic squares are explained.

OPERATIONS ON MAGIC STARS

By assuming that a group of any integers is filled into each cell of magic squares, the operations below are satisfied [6]:

1. If a same number K is added to (or subtracted from) each element of a magic square of order m with magic constant N, the result is also a magic square, with the line sum increased (or decreased) by three times the number added (or subtracted). Thus, the resulting magic constant is N + mK (or $N - mK$).
2. When each element of magic square with magic constant N is multiplied (or divided) by the same number, the result is also a magic square, with the line sum multiplied (or divided) by that number. Thus, the resulting magic constant is $N \times K$ (or $N \div K$)
3. If two (or more) magic squares with magic constant N_1 and N_2 are superimposed, the result is also a magic square, with the line sum equal to the sum of the other two. Thus, the resulting magic constant is $N_1 + N_2$.

PROPERTIES OF MAGIC SQUARES

Assume that some restrictions are given for a group of numbers to be arranged into each cell of magic square of order 3. In this case, by arranging a group of consecutive integers into each cell, a normal magic square of order 3 is obtained. For example, a solution for normal magic square of order 3 is shown in Figure 2.

4	9	2
3	5	7
8	1	6

Figure 2: Normal magic square of order 3

For this magic constant of 15, magic square of order 3 have eight different solutions, which are shown in Figure 3. Note that the magic square in Figure 3(a) corresponds to the magic square in Figure 2.

(a)

4	9	2
3	5	7
8	1	6

(b)

6	1	8
7	5	3
2	9	4

(c)

2	7	6
9	5	1
4	3	8

(d)

8	3	4
1	5	9
6	7	2

(e)

2	9	4
7	5	3
6	1	8

(f)

8	1	6
3	5	7
4	9	2

(g)

6	7	2
1	5	9
8	3	4

(h)

4	3	8
9	5	1
2	7	6

Figure 3: The 8 different solutions for magic square of order 3

However, these eight solutions are actually similar and they are isometric. Thus, so the solution for magic constant of 15 is unique if the complement and transformations of rotation and reflection are not neglected.

The complement and transformations of rotation and reflection in making the solution for magic square of order 3 unique are shown in Section 3.1, Section 3.2 and Section 3.3 respectively.

Complement for Magic Squares

A complement of magic square is obtained when each number in the cell of magic square is subtracted from the largest number of the pattern plus one [7]. For normal magic square of order 3, the largest number is 9, which is 3^2. Hence, every cell in magic square of order 3 is subtracted from the number 10. Now, by subtracting the original magic square in Figure 3(a) from the number 10, a complement of magic square is obtained. Figure 4 shows the complement for normal magic square of order 3 obtained from the original magic square, which corresponds to the magic square in Figure 3(b).

10 – 4	10 – 9	10 – 2
10 – 3	10 – 5	10 – 7
10 – 8	10 – 1	10 – 6

=

6	1	8
7	5	3
2	9	4

Figure 4: Complement of magic square

Rotation for Magic Squares

In the fields of geometry and linear algebra, rotation is just a transformation that is performed by 'spinning' the object around a fixed point or in space that describes the motion of a rigid body around a fixed point. This fixed point is also known as the center of rotation. Usually, rotation is done in the anticlockwise direction [8].

By rotating the magic square, a different solution can be obtained if the magic square is rotated with different degree of rotation, such as 90º, 180º and 270º. For example, when the original magic square in Figure 3(a) is rotated, three different solutions of magic squares can be obtained with three different degrees of rotations, namely 90º, 180º and 270º. The solutions obtained by rotation are shown in Figure 5, which correspond to the magic squares in Figures 3(c), (b) and (d) respectively.

2	7	6
9	5	1
4	3	8

Rotation of 90º

6	1	8
7	5	3
2	9	4

Rotation of 180º

8	3	4
1	5	9
6	7	2

Rotation of 270º

Figure 5: Solutions of rotational magic square

Reflection for Magic Squares

Reflection is another type of transformation. It is a transformation that is performed by 'flipping' the body it is transforming. It is a mapping that transforms an object into its mirror image. For example, a reflection of a letter ***p*** in respect to a vertical line would look like ***q*** and by reflection in a horizontal axis would look like ***b***. A reflection is an involution since when applied twice in succession, every geometrical object will restore to its original state [8]. So, a reflection of magic square is an involution since a magic square will restore to its original form after applying reflection twice in succession.

A reflection of magic square is actually isometric to a rotation of magic square. Figure 6 shows the involution of magic square in Figure 3(b), where the middle magic squares correspond to the magic square in Figure 3(f).

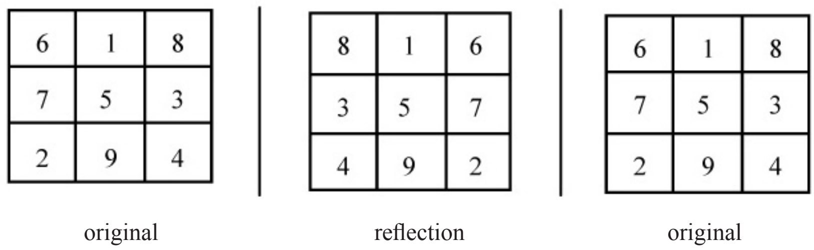

Figure 6: Involution of magic square of order 3

Different solutions of magic square can be obtained after applying reflection to the magic square of order 3. From Figure 3(a), the solution for magic squares of mirror image after reflection is shown in Figure 7, which actually correspond to the magic squares in Figure 3(b), (e), and (f) respectively.

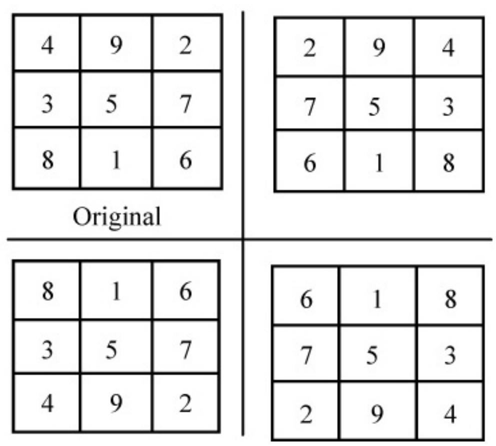

Figure 7: Mirror images of magic square from Figure 3(a)

Now, different solutions of magic square can be obtained after applying reflections to magic square from Figure 3 (c), as shown in Figure 8. The solutions obtained after reflection of magic square from Figure 3 (c) correspond to the magic squares in Figure 3 (h), (g), and (d) respectively.

Thus, the solution of all magic squares of order 3 can be obtained by applying properties such as complement and transformation. These properties of complement and transformation can also be applied to magic square of any order.

2	7	6
9	5	1
4	3	8

Original

6	7	2
1	5	9
8	3	4

4	3	8
9	5	1
2	7	6

8	3	4
1	5	9
6	7	2

Figure 8: Mirror images of magic square from Figure 3(c)

In the next section, methods to generate magic squares of any order manually will be discussed.
Methods to Construct Magic Squares

The orders of magic square are divided into three types, namely odd order magic squares, double-even order magic squares and single-even order magic squares. There are various methods to find the solution for magic square of any order. In this study, the solution for odd order magic square, double-even order magic squares and single-even order magic squares are obtained using De la Laubera's method, Devedec's method and Modified De la Laubera's method respectively [9].

De la Laubera's Method

De la Laubera's method is one of the methods to construct any magic square of odd order, for example magic square of orders 3, 5, 7 and so on.

Now, a solution for magic square of order 3 can be generated by De la Laubera's method using the following steps:

1. Number 1 is placed in the center cell of first row as in Figure 9.

	1	

Figure 9: First step of De la Laubera's method

2. Number 2 is placed into cell after moving one cell to the right and one to above in oblique direction. If there is no cell on the top, then put it on the bottom of column as shown is Figure 10.

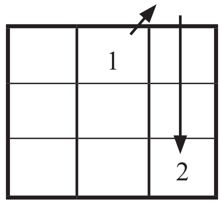

Figure 10: Placing number in opposite side of the column

3. Next, number 3 is put into cell by moving in the diagonal direction to the right. If there is no cell on the right side, then put it on the opposite side of the row as shown in Figure 11.

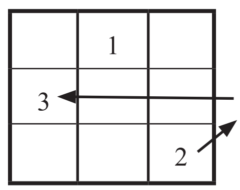

Figure 11: Placing number in opposite side of the row

4. After completing the group of three numbers, one cell is moved down to start generating the next group of three numbers as shown in Figure 12.

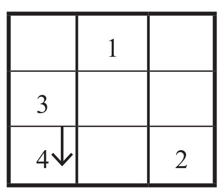

Figure 12: Placing numbers into cell after completing the group of three numbers

5. Steps 2 to 4 are repeated by placing the consecutive numbers into each cell until the third group of three numbers are filled in. The last number of magic squares of any orders is occupied into the cell of the bottom row.

Thus, the solution for magic squares of order 3 is obtained using De la Laubera's method. This solution corresponds to the magic square in Figure 3(b).

Devedec's Method

Devedec's method is one of the methods to construct double-even order magic square. Double-even order magic square is the magic squares of order 4*m*, where *m* is positive integer numbers. Double-even order magic squares are the squares having a multiple of 4 cells, for example magic squares of order 4, 8, 12 and so on.

Considering the magic square of order 4, the solution of magic square of order 4 can be generated using Devedec's method following the steps below:

1. Each main diagonal cell of magic square of order 4 is filled with the X's as shown in Figure13.

X			X
	X	X	
	X	X	
X			X

Figure 13: Magic square filled with Xs

2. A number is inserted into each cell without X, starting from the upper left cell of square and moving towards right. The same procedure is made following row by row. The numbers inserted are the consecutive numbers starting from the count increases by one for each move across the cells until the last cell of the lowest row. The cells occupied X are skipped. The situation is shown in Figure 14.

X	2	3	X
5	X	X	8
9	X	X	12
X	14	15	X

Figure 14: Cell without X filled with consecutive numbers

3. Now, for the cells occupied with X, a number is inserted into the cells starting from the upper left cell of square moving towards right, following row by row. The cells that are not occupied with X are skipped. The numbers inserted are the consecutive numbers starting from $4m$ that is number 16 for this case. The count decrease by one for each move across the cells until last cell of the lowest row as shown in Figure 15.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Figure 15: Replacing X with consecutive numbers while skipping cells without X

Thus, the solution for magic square of order 4 obtained by Devedec' method is shown in Figure 15.

Therefore, magic squares of double-even order can be constructed using Devedec' method. For magic squares of single-even order, one of the methods to construct the solution is by using Modified De la Laubera's method as discussed in Section 4.3.

Modified De la Laubera's Method

Single-even order magic squares are magic squares of even order, i.e. 2 (2*m* + 1) where *m* = {1, 2, 3, 4, 5, …}. For instance, magic squares of order 6 and 10 are two examples of single-even order magic squares. One of the methods to construct single-even order magic square is by Modified De la Laubera's method.

Single-even order magic squares are more complicated to be constructed compared to the other two types of magic squares. Considering the magic square of order 6, the solution of magic square of order 6 can be generated using the following algorithm:

1. A single-even order magic square is divided into four small magic square of order 2*m* + 1. For magic square of order 6, the value of *m* is 1. Thus, magic squares of order 6 is divided into four small magic squares of order 3 and are labeled as squares *a*, *b*, *c*, *d* as shown in Figure 16.

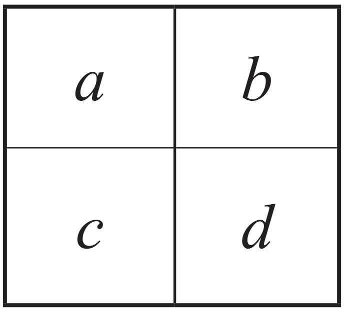

Figure 16: Magic square of order 6 being divided into four small magic square of order 3

2. Beginning with square *a* and starting with consecutive number from 1, magic square *a* is generated using De la Laubera's method as shown in Figure 17.

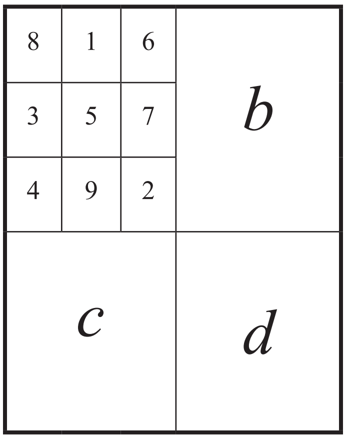

Figure 17: Generating magic square *a*

3. Next, the solution of magic square *d* is generated using De la Laubera's method starting with consecutive number from 10 as shown in Figure 18.

8	1	6			
3	5	7		*b*	
4	9	2			
			17	10	15
	c		12	14	16
			13	18	11

Figure 18: Generating magic square *d*

4. Then, magic square *b* is generated starting from number 19 and magic square *c* is generated starting from number 28, as shown in Figure 19.

8	1	6	26	19	24
3	5	7	21	23	25
4	9	2	22	27	20
35	28	33	17	14	15
30	32	34	12	14	16
31	36	29	13	18	11

Figure 19: Generating magic square *b* and *c*

5. Lastly, transpose the number:

5 and 32,
8 and 35,
4 and 31.

The solution for magic square of order 6 is thus shown in Figure 20.

35	1	6	26	19	24
3	32	7	21	23	25
31	9	2	22	27	20
8	28	33	17	10	15
30	5	34	12	14	16
4	36	29	13	18	11

Figure 20: Solution of magic square order 6

Hence, a solution for magic square of order 6 is obtained using Modified De la Laubera's method.

THE METHOD TO OBTAIN NORMAL MAGIC SQUARES OF ANY ORDER USING MATLAB PROGRAMMING

In MATLAB programming, there is actually a built-in function that can construct magic squares of order $m \times m$. Thus, magic squares can be constructed using commands in MATLAB programming. The method to obtain magic square and their properties using MATLAB will be discussed in the following.

To obtain magic square of any order using MATLAB programming, the method is by writing the command *magic* in the command window. For example, in order to obtain the magic square of order 3, the command is

MS = magic (3).

The output is

```
MS =
        8   1   6
        3   5   7
        4   9   2.
```

The reflection of magic square can also been obtained using MATLAB programming. For example, reflections for magic squares of order 3 in horizontal can be obtained by swapping column 1 and 3. Using MATLAB programming, column 1 and 3 can be swapped with the command *MS (: ,[3 2 1]).* This means that, for each row of matrix *MS*, the elements are reordered in the order 3, 2, 1. For example using *MS=magic (3),* the command to reorder the row in the order 3, 2, 1 is *MS (: ,[3 2 1])*.

The output is

```
ans =
        6   1   8
        7   5   3
        2   9   4.
```

The rotation for magic square of order 3 can also be obtained using MATLAB programming. For instance, the rotation for magic square of order 3 in 90^o can be obtained by transposing the reflections of magic squares of order 3 in horizontal plane using the command " ' ". For example, by using the command *MS (: ,[3 2 1])',*
the output is

```
ans =
      6   7   2
      1   5   9
      8   3   4.
```

The solutions for magic square of order 4 and 6 obtained by the methods discussed above are also verified using MATLAB. The magic square of order 4 and 6 are generated by command *MS4 = magic (4)* and *MS6 = magic (6)* respectively, the output of MATLAB programming for magic squares of order 4 and 6 are displayed as follows:

```
MS4 =
      16   2   3  13
       5  11  10   8
       9   7   6  12
       4  14  15   1,
```

and

```
MS6 =
      35   1   6  26  19  24
       3  32   7  21  23  25
      31   9   2  22  27  20
       8  28  33  17  10  15
      30   5  34  12  14  16
       4  36  29  13  18  11.
```

CONCLUSION

Magic square is one of the different arrays in magic number arrangements, which is arranged in the shape of a square. In this paper, some operations and properties (including the complement, rotation and reflection) of magic square have been presented. Some methods to obtain the solution for magic squares of odd order, single-even order and double-even order have also been discussed. Among the methods are De la Laubera's Method, Devedec's method and also modified De la Laubera's method, respectively. Furthermore, the method to obtain the solutions for magic squares of any order is presented using MATLAB Programming software.

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