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The Sylvester Methods of Constructing Resultant Matrices

(Kepentingan Menterjemah Karya Matematik Lama)

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ABSTRACT

In this paper we make some observations on the classical and modern approaches that can be used for the construction of Sylvester-type resultant matrices. In the modern approach of subdivision based, the matrix construction of a Sylvester-type sparse resultant is based on a subdivision of the Minskowski sum of convex polytopes *Q*, which is perturbed by a small integer δ. A monomial multiple of some polynomial in the system is then associated to each integer point in $\mathring{a} = (Q +) \cap Z^2$, which make up the entries of the desired resultant matrix; thus the matrix and its size depends on δ and the smallest possible resultant matrix can be obtained provided appropriate choice of δ is used. While in the classical approach of constructing a Sylvester matrix, by homogenizing variables leads to a fixed Sylvester resultant matrix of size $S_0 \cup \ldots \cup S_n = N = \binom{d+n}{n}$

Keywords: Sylvester-type Formulae, Sylvester Matrix, Optimal Resultant Matrix

INTRODUCTION

It is known that the resultants eliminate several variables simultaneously and reduce system solving over the complex numbers to univariate polynomial factorization or an eigenproblem, and this fact of resultants' behavior appeared in many research papers, for instance in (Emiris & Mantzaflaris 2009), (Emiris 2005), (D'Andrea & Emiris 2001), (D'Andrea 2002), and (Canny & Emiris 2000). An elementary of resultant theory can be found in (Cox, Little, & O'Shea 2005 chap. 3 & 7). When finding the resultant of a polynomial system from the resultant matrix, the smallest possible resultant matrix is sought and the matrix is optimal when its determinant is precisely equals the resultant. There are two main classes of matrices, namely Sylvester's and Bézout's. The Sylvester methods can be categorized into two approaches. The first is the classical Sylvester matrix approach which only considers multihomogeneous systems obtained by introducing homogenizing variables that homogenize each equation of the system. The second is the modern approach of sparse resultant which generalizes the coefficient matrix of a linear system via the Sylvester's matrix and Macaulay's matrix. Furthermore the sparse resultant matrix approach can be applied to any generic polynomial system without having to introduce homogenizing variables.

In this paper we make some observations on these two approaches used for the construction of Sylvester-type resultant matrices. In the modern approach of subdivision based, the matrix construction of a Sylvester-type sparse resultant is based on a subdivision of the Minskowski sum of convex polytopes *Q*, which is perturbed by a small integer δ. A monomial multiple of some polynomial in the system is then associated to each integer point in $\varepsilon = (Q + \delta) \cap Z^2$, which make up the entries of the desired resultant matrix; thus the matrix and its size depends on δ and the smallest possible resultant matrix can be obtained provided appropriate choice of δ is used. While in the classical approach of constructing a Sylvester matrix, by homogenizing variables leads to a fixed Sylvester resultant matrix of size $S_0 \cup \ldots \cup S_n = N = \binom{d+n}{n}$. Therefore the research pays attention to the sets of monomials' computation associated with the mutually disjoint sets of S_0. ..., S_n with the properties that enable the construction of the resultant matrix.

RESULTANT MATRICES

Given $n+1$ homogeneous polynomials $F_0, \ldots F_n \in \mathbb{C}[x_0, \ldots, x_n]$, the research is motivated to further investigate and apply the conditions on the coefficients of F_0, ..., F_n, in particular the properties related to resultant, such that F_0, ..., F_n, = 0 has a nontrivial solution.

Different resultants exist depending on the space of the roots one wish to characterize, namely projective, or affine. Projective resultants (known as classical) were, namely the the first to be studied, characterizes the roots in projective space. There are many ways to express resultants, which are distinguished into Sylvester, Bezout and hybrid-type formulae (Dickenstein & Emiris 2003), (D'Andrea & Emiris 2001). This paper focusses in discussing the Sylvester-type matrix formulae, which generalizes the coefficient matrix of a linear system and is a generalization of the Macaulay's matrix. Some notations and basic properties for homogeneous polynomials are introduced preceded by the classical construction given in the following section.

The Classical Sylvester Matrix

In this section we gather some basic facts on resultants for homogeneous form in several variables. Although they are very classical objects of study, many fundamental questions about them still remain open.
Let

$$F_i = \sum_{|\alpha| = d_i} u_{i,\alpha} x^{\alpha}, \qquad i = 0, \ldots, n \tag{1}$$

be $n+1$ homogeneous polynomials with real coefficients in $n+1$ variables of degree d_0, ..., d_n , where for each monomial of degree d_i , its coefficients are the variables $u_{i,\alpha}$. In order to describe a polynomial in the coefficients of the F_i, each variable of $u_{i,\alpha}$ is replaced with the corresponding coefficient $c_{i,\alpha}$ when F_i is evaluated at ($c_{i,\alpha}$). We give the following theorem that describes the degree of a resultant.

Theorem 2.1.1 The resultant $R(F_0, F_1, ..., F_n)$ is a homogeneous polynomial in the coefficients of each form F_1 of degree $d_0\, d_1 \ldots d_{i-1} d_{i+1} \ldots d_n$.

This theorem has appeared in several references, for instance in (Gelfand, Kapranov & Zelevinsky 2008 chap. 13) and (Sturmfels 1998). The generalization of this theorem is the theorems on mixed volumes that can be referred to in the same references and in (Emiris 2005). A practical algorithm for computing mixed volumes is presented in (Emiris & Canny 1995).

Consider the system

$$F_0 = \cdots = F_n = 0 \tag{2}$$

In general, for most values of the coefficients of F_i, there are no nontrivial solutions to the system in the complex projective space $P_n(C)$ or P^n, having homogeneous coordinates x, while for certain special values, they exist. In fact the condition for the existence of a solution for system in P^n is a condition on the coefficients of the F_i's. If the coefficients are undetermined variables, then there is a polynomial *R,* known as the *multipolynomial resultant* for the system, in the coefficients of F_i's which vanishes if and only if the system has a complex solution in P^n.

Let $F_0, \ldots, F_n \in \mathbb{C}[x_0, \ldots, x_n]$ be of total degrees d_0, ..., d_n respectively. Set the integer

$$d = 1 + \sum_{i=0}^{n} (d_i - 1) = \sum_{i=0}^{n} d_i - n \tag{3}$$

This is the smallest integer such that every monomial x^a of degree d $=|\alpha|$ divisible by $x_i^{d_i}$ for at least one i. This integer d is called the *critical degree* (Sturmfels 1998). Consequently, this suggests the definition of S_i, $i = 0,..., n$ for the set of monomials of degree d.

Let $V(F) \subset P^n$ be a projective variety which is the set of points of P^n where F vanishes, determined by the nontrivial solutions of (2). The set of monomials in x of degree exactly d has dimension $N = \binom{d+n}{n} = \binom{\sum d_i}{n}$ which is known as *binomial coefficient*. The N basis elements may be ordered in reverse lexicographical order, with x_n^d first, next $x_n^{d-1}x_{n-1}$, etc. This is the default in *Macaulay2* package, since it can be shown that, in many cases, this ordering is theoretically the most efficient (Eisenbud, Grayson & Stillman 1998). In fact the coefficients of F_i's depend on the order of the polynomials. There are many different orderings of the variables for which x_i is last. To organize these N monomials more systematically, which are applicable in forming the multipolynomial resultant of the system (2), N monomials are partitioned into n+1 sets S_0, ..., S_n as defined below:

$$S_0 = \{x^\alpha : |\alpha| = d,\ x_0^{d_0} \text{ divides } x^\alpha\}$$

$$S_1 = \{x^\alpha : |\alpha| = d,\ x_0^{d_0} \text{ doesn't divide } x^\alpha \text{ but } x_1^{d_1} \text{ does}\}$$

$$S_n = \{x^\alpha : |\alpha| = d,\ x_0^{d_0}, \ldots, x_{n-1}^{d_{n-1}} \text{ don't divide } x^\alpha \text{ but } x_n^{d_n} \text{ does}\}.$$

For each i, let X_i be the monomials in S_i divided by $x_i^{d_i}$. The elements of X_i are certain monomials of degree $d - d_i$. The set X_i is called the *multiplier set* and these sets may not be disjoint. Suppose a monomial $x^\alpha = x_0^{a_0} \cdots x_n^{a_n}$ is an element of S_i and $\frac{x^\alpha}{x_i^{d_i}}$ be the corresponding element of X_i. The formula of degree d, shows that the S_i are disjoint, and their union $S_0 \cup \ldots \cup S_n$ is the entire of N. Furthermore the cardinality of each S_i can be determined.

Lemma 2.1.2. *The cardinality of* S_n *equals* $d_0 d_1$, ..., d_{n-1}.

This lemma is derived from (Sturmfels 1998). We use this lemma to count the number of elements in the subset S_n. For instance, in the case $n = 2$, with variables, x, y, z; the degree of each homogeneous polynomial $d_0 = d_1 = 1$ and $d_2 = 2$, letting z last means that S_2 consists of $d_0 \,.\, d_1 = 1.1$ monomial. But if we let x last, it means that S_0 consists of two monomials. Thus if we fix i between 0 and $n - 1$ and order the variables so that x_i came last, then we get a slightly different sets S_0,..., S_n as well as equations (4) below. In particular, any variable x_i can play the role of the last variable x_n.

We have described all the components needed in constructing the resultant matrix M, whose determinant is a multiple of the resultant of the system given as (2). Next we describe the construction of this N x N matrix. Consider the equations

$$\frac{x^\alpha}{x_i^{d_i}} \cdot F_i = 0 \quad \text{for all } x^\alpha \in S_i,\ i = 0, ..., n. \tag{4}$$

As described earlier $\frac{x^{\alpha}}{x_i^{d_i}}$ has degree $d - d_i$ and F_i has degree d_i, it follows that (4) has degree d. Thus each equation of (4) can be written as a linear combination of the N monomials given above. That is writing this equation in terms of the basis vectors yields a row vector of coefficients of length N which form one row of the matrix M. By repeating this process for each element of each S_i, gives the N rows of M. This matrix M is known as *multipolynomial resultant matrix* of system (2) and it is a square matrix. Since $n = 2$, this matrix is also called *Sylvester matrix* for the resultant of two homogeneous polynomials in two variables (or equivalently, two inhomogeneous polynomials in one variable). To illustrate the process, we use a small example with three equations F_0, F_1, and F_2 of degree 1, 1, 2 respectively in three variables x, y, and z (Cox, Little, & O'Shea 2005, chap. 3, p. 104) to compute the sets S_i and the multiplier sets for respectively, taking z last.

Example 2.1. 1.

$$F_0 = a_1x + a_2y + a_3z = 0$$
$$F_1 = b_1x + b_2y + b_3z = 0$$
$$F_2 = c_1x^2 + c_2y^2 + c_3z^2 + c_4xy + c_5xz + c_6yz = 0.$$

Each of these homogeneous polynomials has degree $d_0 = 1$, $d_1 = 1$, and $d_2 = 2$. Total degree of each S_i is $d = 2$ calculated by using formula (3); therefore $N = 6$. The reverse lexicographical order on these 6 monomials of degree 2 in x, y, and z is given by the set $\{x^2, y^2, z^2, xy, xz, yz\}$. These monomials are divided into n+1 sets to list all the elements in each S_i. The corresponding multiplier sets X_i with degree d-d_i are then obtained when x_n^d divides each elements of S_i. The sets S_i and X_i for this example is given by $S_0 = \{x^2, xy, xz\}$, $X_0 = \{x, y, z\}$; $S_1 = \{y^2, yz\}$ $X_1 = \{y, z\}$ and $S_2 = \{z^2\}$, X_2 respectively.

The multipolynomial matrix is then formed by multiplying each element of X_i by F_i and writing the coefficients out in reverse lexicographical order. The columns correspond to the monomials x^2, y^2, z^2, xy, xz, yz and the rows correspond to the six equations $xF_0 = yF_0 = zF_0 = yF_1 = zF_1 = 1F_0 = 0$. Therefore equation (4) translates into the following Sylvester matrix,

$$\begin{pmatrix} a_1 & 0 & 0 & a_2 & a_3 & 0 \\ 0 & a_2 & 0 & a_1 & 0 & a_3 \\ 0 & 0 & a_3 & 0 & a_1 & a_2 \\ 0 & b_2 & 0 & b_1 & 0 & b_3 \\ 0 & 0 & b_3 & 0 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{pmatrix}. \tag{5}$$

The determinant of this matrix; R detects the existence of a nonzero solution to the system (2) in P^2. If the system has a common solution in P^2, R must be zero. In general, the number of nonzero entries in each row of the matrix is equal to the number of terms in the corresponding F_i and this matrix is always sparse. It is the degree of determinant M and must be $\geq \deg_{fi} R$.

$$\begin{aligned} R = {} & a_1^2b_2^2c_3 - a_1^2b_2b_3c_6 + a_1^2b_3^2c_2 - 2a_1a_2b_1b_2c_3 + a_1a_2b_1b_3c_6 + a_1a_2b_2b_3c_5 - a_1a_2b_3^2c_4 \\ & + a_1a_3b_1b_2c_6 - 2a_1a_3b_1b_3c_2 - a_1a_3b_2^2c_5 + a_1a_3b_2b_3c_4 + a_2^2b_1^2c_3 - a_2^2b_1b_3c_5 + a_2^2b_3^2c_1 \\ & - a_2a_3b_1^2c_6 + a_2a_3b_1b_2c_5 + a_2a_3b_1b_3c_4 - 2a_2a_3b_2b_3c_1 + a_3^2b_1^2c_2 - a_3^2b_1b_2c_4 + a_3^2b_2^2c_1. \end{aligned} \tag{6}$$

The Modern Sylvester Matrix

The determinantal formulae for constructing sparse resultant matrix can be found in (Dickenstein & Emiris 2003), (Sturmfels & Zelevinsky 1994), (Weyman & Zelevinsky 1994), (D'Andrea & Dickenstein 2000) and (Emiris & Mantzaflaris 2009). The best one can hope for is a Sylvester-type formula that is a square matrix whose nonzero entries are the coefficients of the given equation and whose determinant equals precisely the resultant as in (Weyman & Zelevinsky 1994) or at least smallest possible square matrix whose determinant is a nonzero multiple of the resultant as presented in (Emiris & Canny 1995).

It is known that most systems of polynomial equations encountered in real world applications are sparse in the sense that only a few monomials appear with nonzero coefficient. Therefore the classical definition of resultant is not well suited to this situation. We will use the same example throughout in this paper to illustrate the modern approach.

Suppose the exponent sets of A_0, ..., A_n of Z^n and the Newton polytopes are the sets Q_i = Conv(A_i) and coherent mixed subdivision of $Q = Q_0 + L + Q_n$. The following process in computing the sparse resultant are adapted from (in (Emiris 2005) and (Canny & Emiris 2000).

1. Fix a set of monomials or exponents, $\varepsilon = Z^n \cap (Q + \delta)$, $\delta \in R^n$, is a small vector chosen so that for every $\alpha \in \varepsilon$ there is a cell R of the mixed subdivision such thatlies in the interior of R + δ.

2. Lift the polytopes $Q_1, \dots, Q_n \subset R^n$ to R^{n+1} by picking random vectors $l_1, \dots, l_n \in Z^n$, and consider the polytopes $Q_i = \{(v, l_i \cdot v: v \in Q_i\} \subset R^n \times R = R^{n+1}$, regard l_i as the linear map $R^n \to R$ defined by $v \to l_i \circ v$, then is the position of the graph of l_i lying over Q_i

Using example 2.2.1, we dehomogenize F_i by letting z = 1 (because z is ordering last),

$$f_0 = a_1 x + a_2 y + a_3 = 0, f_1 = b_1 x + b_2 y + b_3 = 0, f_2 = c_1 x^2 + c_2 y^2 + c_3 + c_4 xy + c_5 x + c_6 y = 0 \qquad (7)$$

The set of exponents (also called suports) appearing in (7),

$$A_0 = A_1 = \{(1,0),(0,1),(0,0)\}, \quad A_2 = \{(2,0),(0,2),(0,0),(1,1),(1,0),(0,1)\}$$

and the Newton polytopes, Q is the convex hull of the supports A_i as in Figure 1. Now, let $m_i = |A_i|$; we have n = 2, m_1 = m_2 = 3, m_3 = 6. To obtain the coherent mixed subdivision of Q, the Newton polytopes are lifted by picking random vectors $l_1, \dots, l_n \in Z^n$, so that $Q_i = \{(v, l_i \cdot v: v \in Q_i\} \subset R^n \times R = R^{n+1}$. In this example, the Newton polytope is, $Q = \{(4,0),(2,2),(2,0),(3,1),(3,0),(2,1),(1,3),(1,1),(1,2)(1,0),(0,4),(0,2),(0,3),(0,1),(0,0)\}$ with the lifting vectors l_0 = (0, 4), l_1 = (2, 1) and l_2 = (5, 7) giving the lifted polytope

$$Q = \begin{Bmatrix} (4,0,12),(2,2,13),(2,0,2),(3,1,11),(3,0,7),(2,1,6),(1,3,15),(1,1,1),(1,2,8)(1,0,2), \\ (0,4,19),(0,2,5),(0,3,12),(0,1,1),(0,0,0) \end{Bmatrix},$$

as in Figure 2.

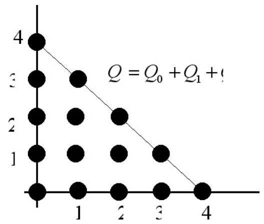

Figure 1: Newton polytopes, Q_i = Conv(A_i)

Note that each point $\hat{Q}$ lies over a point of Q. Figure 2(a) shows the coherent mixed subdivision of Q after the projection $R^3 \to R^2$ onto the first two coordinates carries the lower facets $F \subset \hat{Q}$ onto 2-dimensional polytopes $R \subset Q = Q_0 + Q1 + Q_2$ (there are 6 of them), while Figure 2(b) shows the interior points ε, contains six exponent vectors indicated by dots and as consisting of the monomials $\varepsilon = \mathrm{Z}^n \cap (Q + \delta) = \{x^3y, x^2y^2, x^2y, xy^3, xy^2, xy\}$. This construction was given by (Canny and Emiris, 1993) and some nice pictures appear in (Canny and Emiris, 2000).

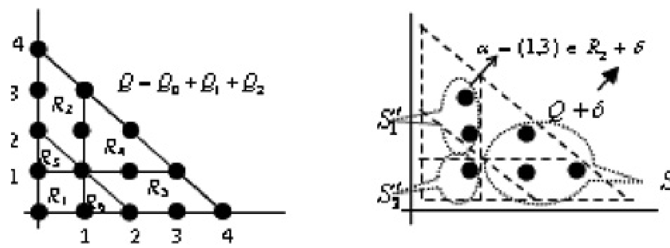

Figure 2: A coherent mixed subdivision and its perturbation ($\delta = (e, e)$)

Definition 2.2.1. A subset of is $S_i \subset \varepsilon$

$S_i = \{\alpha \in \varepsilon$: if $\alpha \in R + \delta$ and $R = F_{0} + ... + F_n$ *is coherent*, *then i is the smallest index such that* F_i *is a vertex*$\}$

This definition gives a disjoint union of the monomials $\varepsilon = S_0 \cup \cdots \cup S_n$. Let v ($\alpha$) denote the vertex, so $F_i = \{\ v(\alpha)\}$, and $v(\alpha) \in A_i$ because $Q_i = \mathrm{Conv}(A_i)$. For instance, the exponent vector α= (1,3)of xy^3 lies in $R_2 + \delta$, see Figure 2(b). If we choose δ = (-e, -e), ε consists of ten exponent vectors or equivalently the ten monomials $x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y, 1$; thus different δ's can give different ε's.

If we denote the resulting set of monomials as S_i', then $S_0' \cup \cdots \cup S_n'$ consists of all monomials of total degree $\leq d$ in $x_1, \ldots, x_n$. The degree of each monomial in S_i' is $\leq d = d_0 + d_1 + d_2 = 4$. Furthermore we see that S_n' consists of the $d_0 + d_{n-1}$ monomials. Our emphasis is on S_n', we will use x^a to denote elements of S_n' and x^β to denote elements of $S_0' \cup \cdots \cup S_{n-1}'$. Then observe that if $x^\alpha \in S_n'$, then has x^a degree $\leq d$-1. If $x^\beta \in S_i'$, $i = 0, \ldots, n-1$ then $x^\alpha / x_i^{v(\alpha)}$ has degree $\leq d$-d. This gives $\deg(X_2') \leq 2$, $\deg(X_1') \leq 3$ and $\deg(X_0') \leq 3$. Therefore we obtain $S_0' = \{x^3y, x^2y^2, x^2y\}$, $X_0' = \{x^2y, xy^2, xy\}$; $S_1' = \{xy^3, xy^2\}$, $X_1' = \{xy^2, xy\}$ and $S_2' = \{xy\}$, $X_2' = \{xy\}$, which form the six equations $x^2y \cdot f_0 = xy^2 \cdot f_0 = xy \cdot f_0 = xy^2 \cdot f_1 = xy \cdot f_1 = xy \cdot f_2 = 0$. Taking all the monomials in S_i' as columns and the six equations as rows, gives the same matrix D_2 as in (5), thus giving $\mathrm{Res}_{1,1,2}(F_0, F_1, F_2) = \mathrm{Res}_{A_0, A_1, A_2}(f_0, f_1, f_2)$. This gives a close relation between the determinant D_2 and $\mathrm{Res}_{A_0, A_1, A_2}$. In general, the exact number of elements in S_n can be determined using the following theorem:

Theorem 2.2.2 The degree of D_n as a polynomial in the coefficients of f_n is the mixed volume $MV_n(Q_0, \ldots, Q_{n-1})$.

The proof of the theorem can be found in (Cox, Little & O'Shea 2005 chap. 7).

CONCLUSION

Under a suitable lifting vector and perturbation vector in the modern subdivision based approach we have obtained the same multipolynomial resultant matrix of size 6 by 6 for system given in (7). The sparse resultant (6) is homogeneous of degree equals the mixed volume, that is, $MV(Q_0, Q_1, Q_2) = 5$. More precisely, the mixed volume of $MV_0(Q_1, Q_2) = 2$, $MV_1(Q_0, Q_2) = 2$, and $MV_2(Q_0, Q_1) = 1$ which implies the sparse resultant of (6) is quadratic in (a_0, a_1, a_2) and in (b_0, b_1, b_2) and of degree 1 in (c_0, c_1, c_2) respectively. Furthermore $|S_2| = 1 = MV_2(Q_0, Q_1)$.

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