Menemui Matematik (Discovering Mathematics)
Vol. 34, No. 1: 17 – 24 (2012)

Higher Order Homotopy Taylor-Perturbation Using Start-system for Multiroots Functions (Homotopi Taylor-Pengusikan Peringkat Tinggi Menggunakan Sistem-mula untuk Fungsi Multipunca)

Nor Hanim Abd. Rahman[1], Arsmah Ibrahim[2] & Mohd Idris Jayes[3]

[1]Department of Computer & Mathematical Sciences, Universiti Teknologi MARA (UiTM), 13500 Permatang Pauh, Pulau Pinang, Malaysia.

[2,3]Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA Malaysia (UiTM), 40450 Shah Alam, Selangor, Malaysia.

[1]norhanim@ppinang.uitm.edu.my, [2]arsmah@tmsk.uitm.edu.my, [3]mohdidris@tmsk.uitm.edu.my

ABSTRACT

Solving nonlinear equations numerically has always been relevant to the real problems. New approaches are actively researched and proposed. One of the approaches involves iterative methods closely related to the classical Newton–Raphson method. The main problems in this approach is to determine the best initial value and the convergence rate. This paper proposes a higher order homotopy Taylor-perturbation using start-system (HHTPss) method to solve multiroots functions. The efficiency of the Classical Newton-Raphson (CNR) and HHTPss are compared and analyzed. Numerical examples are given to illustrate and support the suggested algorithmshe results show that HHTPss offers as an alternative and effective way in solving nonlinear equations.

Keywords: Newton-Raphson, Homotopy Taylor-perturbation, Start-system, Nonlinear, Multiroots

ABSTRAK

Penyelesaian persamaan tak linear berangka sentiasa relevan kepada masalah sebenar. Pendekatan baru sentiasa aktif dikaji dan diusulkan. Salah satu pendekatan melibatkan kaedah lelaran yang berkait rapat dengan kaedah klasik Newton-Raphson. Masalah utama dalam pendekatan ini adalah untuk menentukan nilai awal yang terbaik dan kadar penumpuan. Kertaskerja ini mencadangkan kaedah homotopi Taylor-pengusikan peringkat tinggi dengan menggunakan kaedah sistem-mula (HHTPss) untuk menyelesaikan fungsi multipunca. Kecekapan Newton-Raphson klasik (CNR) dan HHTPss dibandingkan, serta dianalisa. Contoh berangka diberikan untuk menggambarkan dan menyokong keputusan-keputusan algoritma yang disyorkan. Keputusan menunjukkan bahawa HHTPss menawarkan sebagai cara alternatif yang berkesan dalam menyelesaikan persamaan tak linear.

Katakunci: Newton-Raphson, Homotopi Taylor-pengusikan, sistem-mula, tak linear, multipunca

INTRODUCTION & BACKGROUND

Nonlinear problems are not new and indeed have been applied in many areas of science and technology. Numerical method appears to provide as an alternative way to solve this problem. Among the numerical methods known for its efficiency and effectiveness is the Newton-Raphson method which is proven for its second order convergence. However there are problems in this method because its effectiveness lies in the accuracy and closeness of the initial value picked at the beginning of the iteration process.

There are also modified Newton methods for problem such as multiroots where a multiplier ,m' representing the highest power of higher order polynomial functions, is added into the original Newton-Raphson iterative method (S.G. Li *et al.*, 2010; Rafiq & Awais, 2008). However, it is effective only for simple nonlinear multiroots problems, but ineffective for more complex equations and higher order homotopy Taylor-perturbation. Besides the easiness of Newton-homotopy, it does not guarantee to converge.

Other new and surprising methods also offer solutions to problems such as the perturbation technique that is based on an assumption that a small parameter must exist in the equation, homotopy method (He, 1999, 2009) and the hybrids as well as robust approaches of these methods (Saeed *et al.,* 2011; Nor Hanim *et al.,* 2011a, 2011b, 2011c, 2011d, 2010; Palancz, 2010; Saeed & Khthr, 2010; Pakdemirli & Boyaci, 2007). Their approaches are iterative, and in many cases appears to be considerably more computational oriented. The idea of these new approaches are somehow simple and proven effective.

This paper presents hybrid methods of higher order Taylor's series, perturbation techniques, homotopy continuation method and the start-system concepts in order to generate faster and more effective ways to solve multiroots of nonlinear problems for $f(x) = 0$

METHODOLOGY

Definition 2.1. We define the nth Taylor polynomial $P_n(x)$ about $x=x_0$ *as,*

$$P_n(x) = f(x - x_0) \cong f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{(f^n(x_0)(x - x_0)^n)}{n!} = \sum_{i=0}^{n} \frac{(f^i(x_0)(x - x_0)^i)}{i!} \quad (1)$$

Definition 2.2. Perturbation expansion is defined as,

$$\rho(x) = \varepsilon^0 x_0 + \varepsilon^1 x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \ldots + \varepsilon^n x_n = \sum_{i=0}^{n} \varepsilon^i x_i \quad (2)$$

where, ε_i are any real values for $i = 0$ to n.

The basic process of higher-order Taylor homotopy perturbation method is to substitute the perturbation techniques (refer to Eq. (2)) into the Taylors series (refer to Eq. (1)) up to the value of 'n' required. The resulting equations are then summarized as in Table 1.

Secondly, replacing the original function, $f(x)$ into the convex homotopy for the function $H(x,\lambda): \Re \times [0,1] \rightarrow \Re$ *as,* $H(x,\lambda) = (1-\lambda)\,p(x) + \lambda q(x) = 0$ where, λ is an embedded parameter, $\lambda \in [0,1]$, $p(x)$ as the start system, as the target system, and. $H(x,0) = p(x)$ & $H(x,1) = q(x) = f(x)$. Here, the stepsize is set to 0.2 and the stop-criteria is set to 10^{-5}.

To determine the initial value x_0 only equate $p(x)$ to zero. The selection of $p(x)$ only requires a part of the original equation which is known to have at least one trivial solution. There are also several other ways to identify a start-system of a linear homotopy as mentioned by Nor Hanim *et al.,* (2010, 2011a, 2011b, 2011c, 2011d) and Palancz *et al.,* (2010). However, in this paper the start-system of the homotopy Taylor-perturbation is defined as follows:

Definition 2.3. We defined the convex homotopy for the function $H^*(x,\lambda):\Re\times[0,1]\rightarrow\Re$ is defined as,

$$H^*(x,\lambda)=(1-\lambda)(x^n-C)+\lambda(x); \tag{3}$$

where, $p(x) = x^n - C;$

n is preferably the highest power of x of a nonlinear function $f(x)$,
C is any real number in $f(x)$.

Hereafter, our discussion will only proceed with the above method, Eq. (3), for its flexibility to choose the start-system functions as well as the initial values used for iteration. Below are the five algorithms written and used in the higher order homotopy Taylor-perturbation using start-system (HHTPss). The algorithms are in its simplest forms and used in the iterations (refer to Table 1).

Table 1: The iteration scheme of the higher order correctional terms of homotopy Taylor-perturbation (HHTP) method with start-system (ss).

Correctional Terms	Higher Order Homotopy Taylor-Perturbation Method (iterative form, x = $x_{(i+1)}$)
1st order	$x_i-\dfrac{H(x_i,\lambda)}{H'(x_i,\lambda)}$
2nd order	$x_i-\dfrac{H(x_i,\lambda)}{H'(x_i,\lambda)}-\dfrac{H''(x_i,\lambda).H^2(x_i,\lambda)}{2H'^3(x_i,\lambda)}$
3rd order	$x_i-\dfrac{H(x_i,\lambda)}{H'(x_i,\lambda)}-\dfrac{H''(x_i,\lambda).H^2(x_i,\lambda)}{2.H'^3(x_i,\lambda)}+H^3(x_i,\lambda).\dfrac{H'(x_i,\lambda).H'''(x_i,\lambda)-3.H''^2(x_i,\lambda)}{6H'^5(x_i,\lambda)}$
4th order	$x_i-\dfrac{H(x_i,\lambda)}{H'(x_i,\lambda)}-\dfrac{H''(x_i,\lambda).H^2(x_i,\lambda)}{2.H'^3(x_i,\lambda)}+H^3(x_i,\lambda).\dfrac{H'(x_i,\lambda).H'''(x_i,\lambda)-3.H''^2(x_i,\lambda)}{6H'^5(x_i,\lambda)}$ $+\dfrac{H''^2(x_i,\lambda).f^4(x_i,\lambda)}{4.H'^6(x_i,\lambda)}$
5th order	$x_i-\dfrac{H(x_i,\lambda)}{H'(x_i,\lambda)}-\dfrac{H''(x_i,\lambda).H^2(x_i,\lambda)}{2.H'^3(x_i,\lambda)}+H^3(x_i,\lambda).\dfrac{H'(x_i,\lambda).H'''(x_i,\lambda)-3.H''^2(x_i,\lambda)}{6H'^5(x_i,\lambda)}$ $+\dfrac{H''^2(x_i,\lambda).f^4(x_i,\lambda)}{4.H'^6(x_i,\lambda)}-\dfrac{H''^2(x_i,\lambda).f5(x_i,\lambda).[f'''(x_i,\lambda)-3.f'(x_i,\lambda)]}{2.H'^8(x_i,\lambda)}$

Source: (Nor Hanim *et al.,* 2011a, 2011b)

NUMERICAL & ILLUSTRATIVE ANALYSIS

The list of the higher order correctional terms homotopy Taylor-perturbation (HHTP) method by using start-system can be referred at Table 1.While in Table 2, it shows the list of nonlinear equations used in applying the higher order homotopy Taylor-perturbation. The actual roots and the homotopy functions are calculated and compared. Most of the algorithms have been simplified into simpler forms and iterations are done using mathematical software *Maple14*. The choice of a suitable is not unique and different choices of $p(x)$ work better for different types of equations.

Figure 1, Figure 2 and Figure 3 illustrate the closeness of the start-system values suggested to the real roots for the functions (2), (3) and (8), respectively (refer to Table 2).

Table 2: Nonlinear equations

Functions	Nonlinears $q(x) = f(x)$	$p(x) = 0$;	initial value, x_0
1	$(x-2)^2.(x^4+6x-40)$	$(x^4 - 40)$	-2.514866859, 2.514866859
2	$(2x^3-2).(\sin(x))$	$(x^3 - 2)$	1.259921050
3	$(x^2-e^x-3x+2).(\cos(x))$	$(x^2 - 3x)$	0.0, 3.0
4	$(e^x+x^4-2)^4$	$(x^{16} - 16)$	-1.189207115, 1.189207115
5	$(2x^2-9)^3.(\ln(x))$	$(x^2 - 9)$	-3.0 , 3.0
6	$(\tan(x)-\tanh(x))^5$	$\tan(x)$	0.0 -2.514866859, 2.514866859*
7	$(x^5-x^3-2)^3$	$(x^5 - 2)$	1.148698355
8	$(e^x-\sin(x))^2$	$\sin(x)$	0.0-3.0*
9	$(xe^x+2^{-x}+2\cos(x)-6)^2$	$\cos(x)$	1.570796327

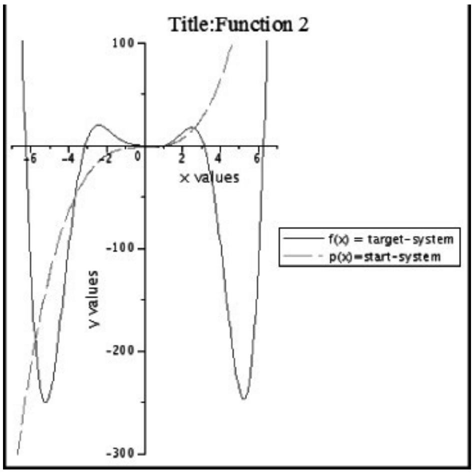

Figure 1: Showing the closeness of roots of the target system and the start-system functions - Function 2

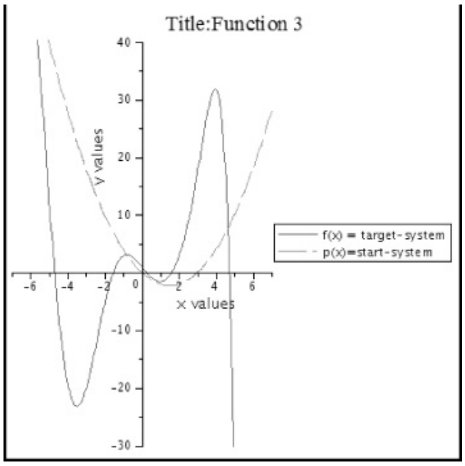

Figure 2: Showing the closeness of roots of the target system and the start-system functions - Function 3

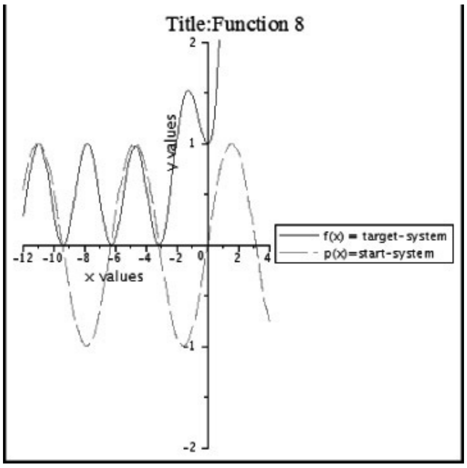

Figure 3: Showing the closeness of roots of the target system and the start-system functions - Function 8

Table 3 shows the efficiency of the iterative hybrid methods from the 1st order to the 5th order homotopy perturbation method using start-system which gives equal or better results in terms of convergence rate as compared to the classical Newton-Raphson. The given test functions (1) - (9) are used and the results of the approximated zeros is given in 10^{-5} error accuracy. The data given in Table 3 are the number of iterations needed in order to converge using CNR and HHTPss. It shows that the computations converge is significantly lower than the classical Newton-Raphsons Method for all of the multiroots functions (1) - (9). The higher the orders of the HHTPss, the faster the converegence rates.

Table 3: The approximated zeros using Classical Newton-Raphson (CNR) and Higher Order Homotopy Taylor-Perturbation (HHTP) using start-system: 1st to 5th Order Correctional Terms on multiroots-functions

NL Functions $q(x) = f(x)=$ target system = 0	Classical Newton-Raphson	HHTPss 1st	HHTPss 2nd.	HHTPss 3rd	HHTPss 4th	HHTPss 5th
$(x-2)^2.(x^4+6x-40)$	6	5	3	3	3	3
$(2x^3-2).(\sin(x))$	4	2	2	2	2	2
$(x^2-e^x-3x+2).(\cos(x))$	3	2	1	1	1	1
$(e^x+x^4-2)^4$	37	34	24	20	20	20
$(2x^2-9)^3.(\ln(x))$	27	21	15	13	11	11
$(\tan(x)-\tanh(x))^5$	118	39	28	24	24	23
$(x^5-x^3-2)^3$	21	19	14	12	12	12
$(e^x-\sin(x))^2$	14	12	9	8	8	8
$(xe^x+2^{-x}+2\cos(x)-6)^2$	3	1	1	1	1	1

CONCLUSIONS

It is thus concluded, method of higher-order homotopy Taylor-perturbation is effective in accelerating the convergence rate not only for simple polynomials but also for the multiroots functions. Combined with the method of start-system, it facilitates the way to determine the initial value for the iteration. Thus, the computing time is reduced.

ACKNOWLEDGEMENT

We would like to thank RMI of Universiti Teknologi MARA Malaysia (UiTM) for the financial supports.

REFERENCES

Chun, C., Bae, H.J. & Neta, B. 2009. New families of nonlinear third-order solvers for finding multiple roots. *Computer and Mathematics with Applications* **58**:1574-1582.

He, J-H. 1999. Homotopy perturbation technique. *Computer Methods in Applied Mechanic and Engineering* **178**:257-262.

He, J-H. 2009. An introduction to the homotopy perturbation method. Computer and Mathematics with applications. **57(3)**:410-412. Doi:10.1016/j.camwa.2008.06.003

Nor Hanim Abd Rahman, Arsmah Ibrahim, Mohd Idris Jayes. 2011a. Higher Order Homotopy Perturbation Method using Start-system in Solving Nonlinear Equations. *Abstract Book: Regional Fundamental Science Congress 2011*. pp.351-352. Faculty of Science, UPM. 2011 ISBN: 978-983-2519-13-3

Nor Hanim Abd Rahman, Arsmah Ibrahim, Mohd Idris Jayes. 2011b. Solving Nonlinear Equations using Improved Higher Order Homotopy Perturbation with start-system. *(CD) Proceeding ICeMATH2011: The International Conference on Numerical Analysis & Optimization*. Universitas Ahmad Dahlan, Yogyakarta, Indonesia (June 6-8, 2011). ISBN: 978-602-98919-1-1

Nor Hanim Abd Rahman, Arsmah Ibrahim, Mohd Idris Jayes. 2011c. Newton-Homotopy Perturbation using start-system for nonlinear (Single, double & triple correctional terms). *International Conference on Mathematical and Computational Biology 2011 (ICMCB2011: Programme Book & Abstracts*. pp54. Universiti Pertanian Malaysia (UPM). April 2011

Nor Hanim Abd Rahman, Arsmah Ibrahim, Mohd Idris Jayes. 2011d. Newton-Homotopy solution for nonlinear equations using Maple14. *Journal of Science and Technology* **3(1).** *UTHM . ISBN: (in press)*

Nor Hanim Abd Rahman, Arsmah Ibrahim, Mohd Idris Jayes. 2010. Newton-Homotopy Solution for Nonlinear Equations using Maple. *Buku Prosiding Jilid 2 Matematik SKASM2010 sempena SKSM18*. ISBN:978-967-5457-44-9.UTHM . Dec 2010.

Palancz, B., Awange, J.L & Lewis, R.H. 2010. Linear Homotopy Solution of Nonlinear Systems of Equations in Geodesy. *J.Geod*. doi 10.1007/s100190-009-0346-x.

Pakdemirli, M. & Boyaci, H. 2007. Generation of root finding algorithms via perturbation theory and some formulas. *Applied Mathematics and Computation*, **184**, 783-788.

Rafiq, A & Awais, M. 2008. Convergence on the Homotopy Continuation Method. *International Journal of Applications Mathematics and Mechanics*, **4(6)**:62-70.

Saeed, R.K. & Khthr, F.W. 2010. Three new iterative methods for solving nonlinear equations. *Australian Journal of Basic & Applied Sciences* **4(6):**1022-1030.

S.G. Li, L.Z. Cheng & B. Neta. 2010. Some fourth-order nonlinear solvers with closed formulae for multiple roots. Computers and Mathematics with Applications **59**:126-135