OMK 2018
ANSWERS (PART A) AND SOLUTIONS (PART B)

CATEGORY: BONGSU

Part A:

1. 30
2. 1035
3. 15
4. 24
5. 45
6. 34

Part B:

1.

Solution: If the third side is $> d$, then it is $\geq d+1$ (since the side lengths are integers), so it is longer than or equal to the other two sides combined, contradicting the triangle inequality.

If the third side is $< d$, then it is $\leq d-1$, so the side with length d is longer than or equal to the other two sides combined, contradicting the triangle inequality.

Hence, the only possibility is that the third side has length d, and the perimeter of the triangle is $2d+1$.

Since the second triangle is equilateral, we require that $2d+1$ be divisible by 3. This implies that $(2d+1)-3 = 2d-2 = 2(d-1)$ is divisible by 3. Since 2 and 3 are coprime, $d-1$ is divisible by 3. Hence the conclusion holds. ■

2.

Solution: If N is even, 9^N ends with 1 and so $9^N - 1$ is divisible by 10. So it suffices to show that the exponent is even, which means that at least one of the terms in the brackets is even.

Suppose that all the 99 terms in the brackets are odd. Then, their sum must be odd. However, their sum is $(a_1+a_2)+\cdots+(a_{98}+a_{99})+(a_{99}+a_1) = 2(a_1+\cdots+a_{99})$ which is even. So at least one bracket is even. ■

Alternative Solution: As before, we show that at least one bracket is even. Assume that each bracket is odd, so it is the sum of an even number and an odd number. This means that $a_1, a_3, \ldots, a_{99}$ are either all odd or all even. In both cases, a_1 and a_{99} share the same parity, which implies that $a_{99}+a_1$ is even.

3.

Solution: First, we start with 2017 + in all possible spaces to get the initial sum 2018, and then we remove as few plus symbols as possible.

If we change $1+1+1+1$ to 1111 (remove 3 consecutive +), then the sum increases by $1111-4=1107$. If we change $1+1+1$ to 111, then the sum increases by $111-3=108$. If we change $1+1$ to 11, then the sum increases by $11-2=9$. These are the three types of changes we can do, as removing more than 3 consecutive + results in a sum which is too large. Suppose we do these three types of changes a, b and c times, respectively.

We want to increase the sum by $8102-2018=6084$. So, we want to minimize $a+b+c$ such that $1107a+108b+9c=6084$. To minimize $a+b+c$, we have to make a as large as possible. Since $\lfloor 6084/1107 \rfloor = 5$, we take $a=5$, and get $1107(5)+108b+9c=6084$, or $108b+9c=549$, which reduces to $12b+c=61$. To minimize $b+c$, we have to make b as large as possible. Since $\lfloor 61/12 \rfloor = 5$, we take $b=5$, which results in $c=61-12(5)=1$.

In short, the minimum value of $a+b+c$ is attained at $a=5$, $b=5$, $c=1$. So we need to remove $3(5)+2(5)+1(1)=26$ plus symbols. Therefore, the answer is $2017-26=1991$. ■

CATEGORY: MUDA

Part A:

1. 9
2. 4
3. 1050
4. 18
5. 9
6. 1

Part B:

1.

Solution: We need $4 \mid 1+2+\cdots+n = \frac{n(n+1)}{2}$, or $8 \mid n(n+1)$. Since n and $n+1$ have different parity, we must have $8 \mid n$ or $8 \mid n+1$. We check that these works: for $n \equiv_8 0$, we may pair every 8 consecutive sticks as follows: $n+(n+7)$, $(n+1)+(n+6)$, $(n+2)+(n+5)$, $(n+3)+(n+4)$ to get four equal lengths. For $n \equiv_8 7$, we start with $7, 1+6, 2+5, 3+4$, and pair every subsequent 8 consecutive sticks as before. There are $252+252 = 504$ such numbers in the range. ■

2.

Solution: Let the last digits be c and d; we write $a = 10m + c$ and $b = 10m + d$, where $c + d = 10$ and $m \leq 1$. Then $ab = (10m + c)(10m + d) = 100m^2 + 10m(c + d) + cd = 100m^2 + 100m + cd = 100m(m+1) + cd$. Since $cd < 100$, the hundreds digit of ab is the last digit of $m(m+1)$, which is even (since one of m and $m+1$ is even). ■

3.

Solution: If $3n+1 = k^2$, then $k \equiv_3 1$ or $k \equiv_3 2$.

If $k = 3m+1$, then $n+1 = 3m^2+2m+1 = m^2+m^2+(m+1)^2$.

If $k = 3m+2$, then $n+1 = 3m^2+4m+2 = m^2+(m+1)^2+(m+1)^2$. ■

CATEGORY: SULONG

Part A:

1. 21
2. 10000
3. 50
4. 45
5. 1275
6. 509

Part B:

1.

Solution: Let the second intersection of AB and Γ_2 be B'. Note that AB' is a diameter, so $\angle ACB' = 90°$. So B' is just the intersection of AB with the line through C perpendicular to AC. Point C' can be defined similarly. As P lies on both Γ_1 and Γ_2, and AC' and AB' are diameters, P is the foot of A altitude in triangle $AB'C'$. Since $\angle ABC' = \angle ACB' = 90°$, we have that the orthocenter of ABC (call it P') lies on Γ, and AP' is its diameter. So the altitude AP passes through the center of Γ. . ■

Alternative Solution: Point P on the circle circumscribed to triangle ACE, so $\angle APC = \angle AEC$ since they are subtended by segment AC. Because of the reflection, we have $\angle AEC = 90° - \angle BAE = 90° - \alpha$.

Analogously, by observing the circle circumscribed to triangle ABD, we can conclude

$$\angle APB = \angle ADB = 90° - \alpha.$$

Therefore, $\angle CPB = \angle APC + \angle APB = 180° - 2\alpha$.

Since $AEPC$ is a cyclic quadrilateral, we have

$$\angle CPE = 180° - \angle CAB - \angle BAE = 180° - 2\alpha.$$

Therefore $\angle CPB = \angle CPE$, so points P, B and E are collinear.

Let A' be the point diametrically opposite point A on the circle circumscribed to triangle ACE. Since AB is the bisector of segment CE, point A' lies on line AB. Therefore, $\angle A'BP = \angle ABE = \beta$. By Thales' theorem we have $\angle APA' = 90°$, so

$$\angle BPA' = 90° - \angle APB = \alpha.$$

Therefore, $\angle AA'P = 180° - \alpha - \beta = \gamma$ and $\angle A'AP = 90° - \angle AA'P = 90° - \gamma$, i.e. $\angle BAP = 90° - \gamma$.

On the other hand, $\angle BOA = 2\gamma$ and $\angle BAO = 90° - \gamma$, so we can conclude that A, O and P lie on the same line. This completes the proof.

2.

Solution: Note that $2018 = 2^4 + 3^4 + 5^4 + 6^4$; write this as $2018 = [2356]$. We note that $6^4 > 2^4 + 3^4 + 4^4 + 5^4$ and $5^4 > 2^4 + 3^4 + 4^4$. So, the first 5 terms in the sequence are $[1234] < [1235] < [1245] < [1345] < [2345]$. The next terms involve 6^4: $[1236] < [1246] < [1346] < [2346] < [1256] < [1356] < [2356]$. Therefore, $i = 12$. ■

3.

Solution: (a) We show that each of the n odd numbers $1, 3, \ldots, 2n-1$ appears exactly once among the n numbers $g_{n+1}, g_{n+2}, \ldots, g_{2n}$. Assume otherwise. Then we have $g_r = g_s$, for some r and s satisfying $n+1 \leq r < s \leq 2n$. This means that $s = 2^N r$ for some $N \geq 1$. However, this is not possible, since $r \geq n+1$ implies $s = 2^N r \geq 2r \geq 2(n+1) > 2n$. Hence our conclusion holds.

Alternative solution for (a): Induct on n. The case $n = 1$ is trivial. Assume that $g_{n+1} + g_{n+2} + \cdots + g_{2n} = n^2$. Then, $g_{n+2} + g_{n+3} + \cdots + g_{2n} + g_{2n+1} + g_{2k+2} = (g_{n+1} + g_{n+2} + g_{n+3} + \cdots + g_{2n}) + g_{2n+1} + g_{2n+2} - g_{n+1} = n^2 + g_{2n+1} + g_{2n+2} - g_{n+1}$. Note that $g_{2n+1} = 2n+1$ and $g_{2n+2} = g_{n+1}$, so the sum equals $n^2 + (2n+1) + 0 = (n+1)^2$, completing the induction.

(b) Using part (a), we have $g_{257} + \cdots + g_{512} = 256^2$, $g_{129} + \cdots + g_{256} = 128^2$, $g_{65} + \cdots + g_{128} = 64^2$, etc. so the answer is $256^2 + 128^2 + 64^2 + 32^2 + 16^2 + 8^2 + 4^2 + 2^2 + 1^2 + 1 = 87382$. ■