OMK 2017
ANSWERS (PART A) AND SOLUTIONS (PART B)

CATEGORY: BONGSU

Part A:

1. 13
2. 20
3. 4
4. 8
5. 122825
6. 13

Part B:

1.

Solution: In a triangle, the sum of two sides is always greater than the other side. So $BC + CA > AB$. If we let L be the area of a triangle, then $L = \frac{h_A \cdot BC}{2} = \frac{h_B \cdot CA}{2} = \frac{h_C \cdot AB}{2}$. Solving for BC, CA, and AB and substituting into the previous inequality, we have $\frac{2L}{h_A} + \frac{2L}{h_B} > \frac{2L}{h_C}$. Dividing through by $2L$ gives the desired result. ■

2.

Solution: (a) 1210, 2020.

(b) 21200. ■

3.

Solution: Using the identities $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ and $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, we have

$$\begin{aligned}
&\frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots + \frac{1}{1+2+3+\ldots+2017} \\
&= \frac{2}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{2}{4 \cdot 5} + \cdots + \frac{2}{2017 \cdot 2018} \\
&= 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{2017} - \frac{1}{2018}\right) \\
&= 2\left(\frac{1}{2} - \frac{1}{2018}\right) = \frac{1008}{1009}.
\end{aligned}$$

So the answer is $\frac{1008}{1009}$. ■

CATEGORY: MUDA

Part A:

1. 30
2. 48
3. 53
4. 7
5. 61
6. 12

Part B:

1.
Solution: Let $\angle A = \theta$. Since $AB = CD$ and $BC = DA$, we only need to prove that $2(AB^2 + BC^2) = AC^2 + BD^2$. Apply the cosine rule to triangles ABC and BCD to get

$$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos(180^\circ - \theta)$$
$$BD^2 = AB^2 + BC^2 - 2(AB)(BC)\cos\theta$$

By adding these two equations and using the fact that $\cos(180^\circ - \theta) = -\cos\theta$, we get the desired result. ■

2.
Solution: Let $N = 2k + 3$ where $k \geq 1$ is an integer. Then

$$N^2 + 5 = (2k+3)^2 + 5 = 4k^2 + 12k + 14 = k^2 + (k+1)^2 + (k+2)^2 + (k+3)^2.$$

Note that all summands are different and positive since $1 \leq k < k+1 < k+2 < k+3$. ■

3.

Solution: First, we prove the identity

$$\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}=1+\frac{1}{k}-\frac{1}{k+1}.$$

Indeed,

$$\begin{aligned}
&\left(1+\frac{1}{k}-\frac{1}{k+1}\right)^2\\
&=1^2+\left(\frac{1}{k}\right)^2+\left(-\frac{1}{k+1}\right)^2+2(1)\left(\frac{1}{k}\right)+2(1)\left(-\frac{1}{k+1}\right)+2\left(\frac{1}{k}\right)\left(-\frac{1}{k+1}\right)\\
&=1+\frac{1}{k^2}+\frac{1}{(k+1)^2}+2\left(\frac{1}{k}-\frac{1}{k+1}-\frac{1}{k(k+1)}\right)\\
&=1+\frac{1}{k^2}+\frac{1}{(k+1)^2},
\end{aligned}$$

due to the fact that $\frac{1}{k}-\frac{1}{k+1}-\frac{1}{k(k+1)}=0.$

Using the identity, we have this telescoping sum:

$$\begin{aligned}
&\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}+\cdots+\sqrt{1+\frac{1}{2016^2}+\frac{1}{2017^2}}\\
&=1+\frac{1}{1}-\frac{1}{2}+1+\frac{1}{2}-\frac{1}{3}+1+\frac{1}{3}-\frac{1}{4}+\cdots+1+\frac{1}{2016}-\frac{1}{2017}\\
&=2017-\frac{1}{2017}\\
&=\frac{4068288}{2017}.
\end{aligned}$$

The answer is $\frac{4068288}{2017}$. ■

CATEGORY: SULONG

Part A:

1. 72
2. 10
3. 340
4. 45
5. 6
6. 7

Part B:

1.

Solution: Let the areas be n, $n+1$, $n+2$, $n+3$. So the area of the quadrilateral is $4n+6$. Triangle CBD is similar to triangle CMN, with ratio 2. Then $|CBD| = 4|CMN|$. Note that $|CMN| \geq n$ by assumption, so we have

$$|ABD| = |ABCD| - |CBD| = |ABCD| - 4|CMN| \leq (4n+6) - 4n = 6.$$

Hence the maximum area is 6. ■

2.

Solution: (a) It suffices to prove that $(2n^2+2n+1)^2 = (2n^2+2n)^2 + (2n+1)^2$. Using the identity $(x+1)^2 = x^2+2x+1$, we have

$$\begin{aligned}(2n^2+2n+1)^2 &= (2n^2+2n)^2 + 2(2n^2+2n) + 1\\ &= (2n^2+2n)^2 + (4n^2+4n+1)\\ &= (2n^2+2n)^2 + (2n+1)^2,\end{aligned}$$

as desired.

(b) Fix a_1 to be any odd number. Obviously a_1^2 is a perfect square. Since a_1 is odd, we have $a_1 = 2n_1+1$ for some integer n_1. Take $a_2 = 2n_1^2 + 2n_1$. By part (a), the sum $a_1^2 + a_2^2 = (2n_1+1)^2 + (2n_1^2+2n_1)^2$ is a perfect square; moreover, it is a square of an odd number. Let the odd number be written as $2n_2+1$, for some integer n_2. Then take $a_3 = 2n_2^2 + 2n_2$. By part (a), the sum $a_1^2 + a_2^2 + a_3^2 = (2n_2+1)^2 + (2n_2^2+2n_2)^2$ is an odd perfect square. Hence we can repeat this process to construct the 2017 a_i's. ■

3.

Solution: The result is trivially true for $n = 1$. Assume that $S_n = n$, and we claim that $S_{n+1} = n + 1$. The nonempty subsets of $\{1, 2, \ldots, n + 1\}$ are of two types: those that do not contain $n + 1$, and those that do. The first type are precisely the nonempty subsets of $\{1, 2, \ldots, n\}$ so their contribution to the sum is $S_n = n$, by assumption. The second type are precisely the sets $S \cup \{n + 1\}$, where S is any subset of $\{1, 2, \ldots, n\}$, including the empty set. The contribution to the sum equals $\frac{1}{n+1}S_n + \frac{1}{n+1} = \frac{n}{n+1} + \frac{1}{n+1} = 1$. Therefore, we have $S_{n+1} = n + 1$, as claimed. ■

Solution 2: If we assume that the empty set has product 1, the sum can be factored as follows:

$$\begin{aligned} S_n + 1 &= \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{n}\right) \\ &= \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{n+1}{n}\right) \\ &= n + 1 \end{aligned}$$

using telescoping product. So $S_n = n$. ■