OMK 2016
ANSWERS (PART A) AND SOLUTIONS (PART B)

CATEGORY: BONGSU

Part A:

1. 36
2. 81
3. 111
4. 300
5. 2044
6. 64

Part B:

1.

Solution: For any rectangle, any straight line through the center divides the rectangle into two regions with equal areas. Using pencil and straightedge, draw the two diagonals of each rectangle. The diagonal meets at the center of the rectangle. Draw a straight line through the centers of both rectangles. This line divides both rectangles into two equal areas, thus it also divides the shaded area into two regions with equal areas.

2.

Solution: If the first number does not end with 9, then the digit sum of the two numbers are consecutive, so they cannot be both divisible by 5. So we choose the first number to end with 9. If the first number is in the form $\cdots A\underbrace{99\cdots 9}_{B\ 9\text{s}}$ (last non-9 digit is A, followed by B 9s), then the second number ends with digit $A+1$ followed by B 0s, other digits unchanged. So the digit sum will decrease by $9B-1$. For both number to be divisible by 5, we need $9B-1$ to be divisible by 5, for example $B=4$. Then we set the other digits to have digit sum divisible by 5. A simple example is 49999 and 50000.

Answers:

All possible answers $N, N+1$ have the form

$$N = [X][Y]$$

where X is a number with digit sum r, and Y is a string of 9s of length s, where $r, s \in \{4, 9, 14, 19, 24, \ldots\}$. For example, this is a valid solution: 77999999999, 78000000000.

3.

Solution: If there are n children, and everyone receive a different number of sweets, then the total number of sweets must be at least $0 + 1 + 2 + \cdots + (n-2) + (n-1) = n(n-1)/2$, which is the sum of the smallest n nonnegative integers. If the number of sweets is $> n(n-1)/2$, we can distribute $0, 1, 2, \ldots, n-2$ to the first $n-1$ children, and pass the rest of the sweets (of which there are $> n-2$) to the last child.

Conversely, if there are less than $n(n-1)/2$ sweets, then some pair of children will receive the same number of sweets. So we want to find the smallest n such that $n(n-1)/2 > 1000$. The answer is $n = 46$.

CATEGORY: MUDA

Part A:

1. 3
2. 7
3. 12
4. 1500
5. 0
6. 9

Part B:

1.

Solution: By Pythagoras theorem applied to triangle FCD, we get $FD = 65$. Let $BY = x$. Triangles DAE and EBF are similar, so $\frac{x+16}{15} = \frac{48}{x}$. This reduces to a quadratic equation in x, with solution $x = 20$. We may use Pythagoras theorem for the rest, obtaining $DE = 39$ and $EF = 52$. So the perimeter is $65 + 39 + 52 = 156$.

2.

Solution: Let the numbers be $n, n+1, n+2, n+3, n+4$ and let $S(x)$ be the digit sum of x. We claim that either n or $n+2$ ends with 9.

If none of the numbers ends with 9 or $n+4$ ends with 9, then $S(n), S(n+1), S(n+2), S(n+3), S(n+4)$ are 5 consecutive integers. Among any 5 consecutive positive integers, at least two of them are even: either both are greater than 2, or we have 1, 2, 3, 4, 5, or we have 2, 3, 4, 5, 6. In all cases, at least two of them are not prime. So this case is impossible.

If $n+3$ ends with 9, then $S(n), S(n+1), S(n+2), S(n+3)$ are 4 consecutive integers with $S(n+3) \geq 9$. So at least 2 of them are even and greater than 2, i.e., not prime. So this case is impossible.

If $n+1$ ends with 9, then $S(n), S(n+1)$ are 2 consecutive integers with $S(n+1) \geq 9$, so one of them is even and greater than 2, i.e., not prime. Among the three consecutive integers $S(n+2), S(n+3), S(n+4)$, one of them is not prime. So this case is impossible.

So either n or $n+2$ ends with 9.

If n ends with 9, we require $S(n+1), S(n+2), S(n+3), S(n+4)$ to have at least 3 primes. The only possibility is 2, 3, 4, 5. We also require $S(n)$ to be prime. A possible answer is $199, 200, 201, 202, 203$.

If $n+2$ ends with 9, we require $S(n), S(n+1), S(n+2)$ to have at least 2 primes, and $S(n+3), S(n+4)$ to be both primes. Therefore $S(n+3) = 2$ and $S(n+4) = 3$. A possible answer is $197, 198, 199, 200, 201$.

Answers:

To check whether an answer is legitimate, check the digit sums. The digit sums must either be

(a) p, 2, 3, 4, 5, where p is a prime, or

(b) p, $p+1$, $p+2$, 2, 3, where p and $p+2$ are both primes.

3.

Solution: The kth triangular number is equal to $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.

(a) Yes, 2016 is the 63rd triangular number, because $\frac{63\times 64}{2} = 2016$.

(b) Let $a = (2r+1)^2$, where r is an integer, so $b = \frac{(2r+1)^2-1}{8} = \frac{r^2+r}{2}$. If n is a triangular number, $n = \frac{k(k+1)}{2}$ for some integer k. Then we have

$$\begin{aligned} an + b &= (2r+1)^2 \frac{k(k+1)}{2} + \frac{r^2+r}{2} \\ &= \frac{4r^2k^2 + 4rk^2 + k^2 + 4r^2k + 4rk + k + r^2 + r}{2} \\ &= \frac{(2rk+k+r)(2rk+k+r+1)}{2}, \end{aligned}$$

which is the $(2rk + k + r)$th triangular number.

CATEGORY: SULONG

Part A:

1. 72
2. 5
3. 13
4. 15
5. 48
6. 31

Part B:

1.

Solution: Let the tangent points of the circle to the sides BC, CD, DA be G, F, E, respectively. Let $a = \angle ODE = \angle ODF$, $b = \angle OCF = \angle OCG$, $c = \angle OAE = \angle OBG$. The sum of angles in the quadrilateral is $2a + 2b + 2c = 360°$, so $a + b + c = 180°$. Hence, $\angle AOD = 180° - a - c = b$ and $\angle BOC = 180° - b - c = a$. Therefore, triangles DOA and OCB are similar, implying $AD/OB = AO/BC$. This reduces to $AO \times OB = AD \times BC$. Plugging $AO = OB = AB/2$ gives the required result.

2.

Solution: Let a and b be the number of digits in the decimal expansions of 2^{2016} and 5^{2016}, respectively. Then

$$10^{a-1} < 2^{2016} < 10^a \qquad \text{and} \qquad 10^{b-1} < 5^{2016} < 10^b.$$

(The strict inequality on the left side is because both 2^{2016} and 5^{2016} are not divisible by 10). Multiplying both inequalities together gives

$$10^{a+b-2} < 10^{2016} < 10^{a+b}.$$

Therefore, $a + b - 2 < 2016 < a + b$. Since there is only one integer strictly between $a + b - 2$ and $a + b$, we deduce that $a + b - 1 = 2016$. The answer is $a + b = 2017$.

3.

Solution: The result is trivial for $n = 2$, we assume henceforth that $n \geq 3$.

For any positive integer k, we have

$$n^k - 1 = (n-1)(n^{k-1} + n^{k-2} + \cdots + n^2 + n + 1).$$

Therefore,

$$n^{n-1} - 1 = (n-1)(n^{n-2} + n^{n-3} + \cdots + n^2 + n + 1).$$

It suffices to show that the second factor $n^{n-2} + n^{n-3} + \cdots + n^2 + n + 1$ is divisible by $n - 1$. We have

$$\begin{aligned}
&n^{n-2} + n^{n-3} + \cdots + n^2 + n + 1 \\
&= (n^{n-2} - 1) + (n^{n-3} - 1) + \cdots + (n^2 - 1) + (n - 1) + (1 - 1) + (\underbrace{1 + 1 + \cdots + 1}_{n-1 \text{ 1s}}) \\
&= (n^{n-2} - 1) + (n^{n-3} - 1) + \cdots + (n^2 - 1) + (n - 1) + (n - 1),
\end{aligned}$$

where we subtract 1 from each term (there are $n - 1$ terms in all) and then add $n - 1$ in the end. Now, all bracketed terms are in the form $n^k - 1$, where $k = 1, 2, \ldots, n - 2$, all of which are divisible by $n - 1$. So we are done.